Vectors, inner products and norms [*]

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First version: December 2019
Revised: December 2019
This version: December 2019
Compiled: February 7, 2023, 16:19

[*] This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), the Toulouse School of Economics (Pierre-de-Fermat Chair of excellence), the Universitad Carlos III de Madrid (Banco Santander de Madrid Chair of excellence), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

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Contents

List of Assumptions, Propositions and Theorems

1. Vector spaces and subspaces

Definition 1.1 SCALAR SET. *A scalar set is a field $\mathbb{F}$ on which operations of addition, subtraction, multiplication, and division are defined.*

The rational numbers $\mathbb{Z}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ are fields. A field has a zero element 0 and a unity element 1.

Definition 1.2 VECTOR SPACE. *A* vector space (*or* linear space) *on a field $\mathbb{F}$ is a set V which satisfies the following axioms.*

(1) *Any two elements x, $y \in V$ determine a unique third element $x + y \in V$, called the* sum *of x and y, such that, for all x, y, $z \in V$,*

- (a) $x + y = y + x\,;$ (*Commutativity of addition*)
- (b) $(x + y) + z = x + (y + z)\,;$ (*Associativity of addition*)
- (c) *there exists* $\mathbf{0} \in V$ *such that* $x + \mathbf{0} = x\,;$ (*Identity element of addition*)
- (d) *there exists* $-x \in V$ *such that* $x + (-x) = \mathbf{0}\,.$ (*Existence of additive inverse*)

(2) *Any two elements $x \in V$ and $\alpha \in \mathbb{F}$ determine a unique element $\alpha x \in V$, called the* product *of α and x, such that, for all $x \in V$ and α, $\beta \in \mathbb{F}$,*

- (a) $\alpha(\beta x) = (\alpha\beta)x\,;$ (*Associativity of scalar multiplication*)
- (b) $1x = x.$ (*Identity element of scalar multiplication*)

(3) *The operations of addition and multiplication satisfy two distributive properties: for all x, $y \in V$, and α, $\beta \in \mathbb{F}$,*

- (a) $(\alpha + \beta)x = \alpha x + \beta x\,;$ (*Distributivity of scalar addition on scalar multiplication*)
- (b) $\alpha(x + y) = \alpha x + \alpha y\,.$ (*Distributivity of scalar multiplication on vector addition*)

Notation 1.1 *To indicate that V is vector space over the field $\mathbb{F}$, we can denote V by $V(\mathbb{F})$.*

Definition 1.3 SUM OF SUBSETS. *Let V be a vector space on a scalar set $\mathbb{F}$, and S_1, S_2 two subsets of V, $x \in V$, and $\alpha \in \mathbb{F}$. Then*

$$S_1 + S_2 := \{y : y = x_1 + x_2,\ x_1 \in S_1,\ x_2 \in S_2\}\,, \tag{1.1}$$

$$x + S_1 := \{x\} + S_1 = \{y : y = x + x_1,\ x_1 \in S_1\}\,, \tag{1.2}$$

$$\alpha S_1 := \{y : y = \alpha x_1,\ x_1 \in S_1\}\,. \tag{1.3}$$

2. Inner-product spaces

For any complex number $z = x_1 + ix_2 \in \mathbb{C}$ (where $x_1, x_2 \in \mathbb{R}$), we denote:

$$\mathrm{Re}(z) := x_1,\ \mathrm{Im}(z) := x_2,\ \bar{z} := x_1 - ix_2\,,\ |z| := \sqrt{x_1^2 + x_2^2}\,. \tag{2.1}$$

$\bar{z} = x_1 - ix_2$ is the *complex conjugate* of z. If $z_1, z_2 \in \mathbb{C}$,

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)\,, \quad \mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)\,. \tag{2.2}$$

Definition 2.1 INNER PRODUCT SPACE. *Let H be a vector space on a scalar set $\mathbb{F}$. An* inner product *on H is an application which associates to each pair of elements x and y in H a scalar $\langle x, y\rangle \in \mathbb{F}$ such that, for all $x,\ y,\ z \in H$,*

(a) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(b) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

(c) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$, *for all* $\alpha \in \mathbb{F}$;

(d) $\langle x, x\rangle \geq 0$;

(e) $\langle x, x\rangle = 0$ *if and only if* $x = 0$.

If $\langle\cdot,\cdot\rangle$ is an inner product on H, the pair $(H, \langle\cdot,\cdot\rangle)$ is called an inner product space*, and the elements of H are also called elements of the inner-product space $(H, \langle\cdot,\cdot\rangle)$. If $\mathbb{F} = \mathbb{R}$, we say that $(H, \langle\cdot,\cdot\rangle)$ is a* real inner-product space*, and if $\mathbb{F} = \mathbb{C}$, we say that $(H, \langle\cdot,\cdot\rangle)$ is a* complex inner-product space*. When there is no ambiguity on the definition of the inner product, the inner-product space $(H, \langle\cdot,\cdot\rangle)$ may simply be denoted H. Unless stated otherwise, we assume that $\mathbb{F} = \mathbb{C}$.*

Remark 2.1 An *inner-product space* is also called a *pre-Hilbert space* [Luenberger (1969, Chapter 3)].

Proposition 2.1 ELEMENTARY PROPERTIES OF INNER PRODUCT. *Let H be an inner-product space, and $x,\ y,\ z \in H$. Then*

(a) $\langle x, y\rangle \langle y, x\rangle = \langle x, y\rangle \overline{\langle x, y\rangle} = |\langle x, y\rangle|^2 = |\langle y, x\rangle|^2$;

(b) $\langle x, \alpha y\rangle = \bar{\alpha} \langle x, y\rangle$, *for all* $\alpha \in \mathbb{F}$;

(c) $\langle \alpha x, \beta y\rangle = \alpha\bar{\beta} \langle x, y\rangle$, *for all* $\alpha,\ \beta \in \mathbb{F}$;

(d) $\langle \alpha x, \alpha y\rangle = |\alpha|^2 \langle x, y\rangle$, *for all* $\alpha \in \mathbb{F}$;

(e) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$, *for all* $\alpha, \beta \in \mathbb{F}$; (*Linearity*)

(f) $\langle z, \alpha x + \beta y\rangle = \bar{\alpha} \langle z, x\rangle + \bar{\beta} \langle z, y\rangle$, *for all* $\alpha, \beta \in \mathbb{F}$;

(*g*) $\mathrm{Re}(\langle \alpha x, y\rangle) = \mathrm{Re}(\alpha)\,\mathrm{Re}(\langle x, y\rangle) - \mathrm{Im}(\alpha)\,\mathrm{Im}(\langle x, y\rangle)$, *for all* $\alpha \in \mathbb{F}$;

(*h*) $\mathrm{Im}(\langle \alpha x, y\rangle) = \mathrm{Re}(\alpha)\,\mathrm{Im}(\langle x, y\rangle) - \mathrm{Im}(\alpha)\,\mathrm{Re}(\langle x, y\rangle)$, *for all* $\alpha \in \mathbb{F}$;

(*i*) $\mathrm{Re}(\langle x, \alpha y\rangle) = \mathrm{Re}(\alpha)\,\mathrm{Re}(\langle x, y\rangle) + \mathrm{Im}(\alpha)\,\mathrm{Im}(\langle x, y\rangle)$, *for all* $\alpha \in \mathbb{F}$;

(*j*) $\mathrm{Im}(\langle x, \alpha y\rangle) = \mathrm{Re}(\alpha)\,\mathrm{Im}(\langle x, y\rangle) + \mathrm{Im}(\alpha)\,\mathrm{Re}(\langle x, y\rangle)$, *for all* $\alpha \in \mathbb{F}$.

Proposition 2.2 ELEMENTARY PROPERTIES OF REAL INNER-PRODUCT VECTOR SPACES. *Let H be an inner-product space with scalar set $\mathbb{F} = \mathbb{R}$, and $x,\ y,\ z \in H$. Then*

(*a*) $\langle x, y\rangle = \langle y, x\rangle$; (*Symmetry*)

(*b*) $\langle x, y\rangle \langle y, x\rangle = \langle x, y\rangle^2 = \langle y, x\rangle^2$;

(*c*) $\langle \alpha x, \beta y\rangle = \alpha\beta \langle x, y\rangle$, *for all* $\alpha,\ \beta \in \mathbb{F}$;

(*d*) $\langle \alpha x, \alpha y\rangle = \alpha^2 \langle x, y\rangle$, *for all* $\alpha \in \mathbb{F}$;

(*e*) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$, *for all* $\alpha, \beta \in \mathbb{F}$; (*Linearity*)

(*f*) $\langle z, \alpha x + \beta y\rangle = \alpha \langle z, x\rangle + \beta \langle z, y\rangle$, *for all* $\alpha, \beta \in \mathbb{F}$;

Definition 2.2 NORM ASSOCIATED WITH AN INNER PRODUCT. *The* norm *of an element x of an inner-product space $(H, \langle\cdot,\cdot\rangle)$ is defined by*

$$\|x\| = \sqrt{\langle x, x\rangle}\ . \tag{2.3}$$

Example 2.1 EUCLIDEAN SPACE. $\mathbb{R}^n$ with the usual scalar product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{2.4}$$

where $x = (x_1, \dots, x_n)'$ and $y = (y_1, \dots, y_n)'$, where $x_i \in \mathbb{R}$ and $y_j \in \mathbb{R}$ for all i and j, is an inner-product space whose norm is the usual Euclidean norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\ . \tag{2.5}$$

Example 2.2 COMPLEX EUCLIDEAN SPACE. $\mathbb{C}^n$ with the scalar product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i \tag{2.6}$$

where $x = (x_1, \dots, x_n)'$ $y = (y_1, \dots, y_n)'$, where $x_i \in \mathbb{C}$ and $y_j \in \mathbb{C}$ for all i and j, is an inner-product space. The associated norm is:

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\ . \tag{2.7}$$

Example 2.3 REAL-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2 = L^2(\Omega, \mathscr{A}, P)$ of all the random variables $X : \Omega \to \mathbb{R}$ such that $E(X^2) < \infty$ with

$$\langle X, Y\rangle = E(XY), \text{ for } X, Y \in L^2, \tag{2.8}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(X^2)]^{1/2}. \tag{2.9}$$

Example 2.4 COMPLEX-VALUED SQUARE INTEGRABLE RANDOM VARIABLES. The set $L^2_{\mathbb{C}}(\Omega, \mathscr{A}, P)$ of all the random variables $X : \Omega \to \mathbb{C}$ such that $E(|X|^2) < \infty$ with

$$\langle X, Y\rangle = E(X\overline{Y}) \tag{2.10}$$

is an inner-product space. The associated norm is:

$$\|X\| = [E(|X|^2)]^{1/2}. \tag{2.11}$$

For many purposes, it is useful to scale $\langle x, y\rangle$ by the norms of x and y. By convention, we set:

$$\frac{\langle x, y\rangle}{\|x\|} = 0 \quad \text{and} \quad \frac{\langle x, y\rangle}{\|x\|^2} = 0 \quad \text{if } \|x\| = 0, \tag{2.12}$$

$$\frac{\langle x, y\rangle}{\|y\|} = 0 \quad \text{and} \quad \frac{\langle x, y\rangle}{\|y\|^2} = 0 \quad \text{if } \|y\| = 0, \tag{2.13}$$

$$\frac{\langle x, y\rangle}{\|x\|\,\|y\|} = 0 \quad \text{if } \|x\| = 0 \text{ or } \|y\| = 0. \tag{2.14}$$

Definition 2.3 H-CORRELATION AND H-REGRESSION. *The H-*correlation *between y and x associated with the inner-product space $(H, \langle\cdot,\cdot\rangle)$ is*

$$\rho_H(x, y) = \frac{\langle x, y\rangle}{\|y\|\,\|x\|}, \quad x, y \in H, \tag{2.15}$$

*and the H-*regression *coefficient of y on x is*

$$\beta_H(x, y) = \frac{\langle x, y\rangle}{\|x\|^2}, \quad x, y \in H. \tag{2.16}$$

$\rho_H(x, y)$ and $\beta_H(x, y)$ may be a complex numbers. In general, we can have $\rho_H(y, x) \neq \rho_H(x, y)$ and $\beta_H(y, x) \neq \beta_H(x, y)$. More explicitly,

$$\rho_H(y, x) = \bar{\rho}_H(x, y), \tag{2.17}$$

$$\beta_H(y, x) = \frac{\langle y, x\rangle}{\|y\|} = \frac{\overline{\langle x, y\rangle}}{\|y\|} = \frac{\|x\|}{\|y\|}\frac{\overline{\langle x, y\rangle}}{\|x\|} = \frac{\|x\|}{\|y\|}\bar{\beta}_H(x, y). \tag{2.18}$$

Definition 2.4 ANGLE AND ORTHOGONALITY. *Let x and y be two elements of a real inner product space $(H, \langle\cdot,\cdot\rangle)$. The* angle *between the vectors x and y is defined by*

$$\theta = \cos^{-1}[\langle x, y\rangle / (\|x\|\,\|y\|)] \,. \tag{2.19}$$

We say that x and y are orthogonal *(denoted $x \perp y$) if and only if*

$$\langle x, y\rangle = 0 \,. \tag{2.20}$$

Definition 2.5 ORTHOGONAL SUBSETS. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner-product space, and S_1, S_2 two subsets of H. Then we say that S_1 and S_2 are orthogonal [denoted $S_1 \perp S_2$] if and only if*

$$[x \in S_1 \text{ and } y \in S_2 \Rightarrow \langle x, y\rangle = 0] \quad \text{for all } x, y \in H. \tag{2.21}$$

Proposition 2.3 BASIC PROPERTIES OF THE NORM. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Then the following properties hold for all $x, y \in H$:*

(*a*) $\|\alpha x\| = |\alpha|\,\|x\|$, *for all* $\alpha \in \mathbb{F}$;

(*b*) $\|x\| \geq 0$;

(*c*) $\|x\| = 0$ *if and only if* $x = 0$;

(*d*) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}[\langle x, y\rangle]$;

(*e*) $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - [\langle x, y\rangle + \langle y, x\rangle] = \|x\|^2 + \|y\|^2 - 2\,\mathrm{Re}[\langle x, y\rangle]$;

(*f*) *if* $\langle x, y\rangle = 0$, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$; (*Pythagorean law*)

(*g*) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow \mathrm{Re}[\langle x, y\rangle] = 0$; (*Generalized Pythagorean law*)

(*h*) $\|x+y\|^2 > \|x\|^2 + \|y\|^2 \Leftrightarrow \mathrm{Re}[\langle x, y\rangle] > 0$;

(*i*) $\|x+y\|^2 < \|x\|^2 + \|y\|^2 \Leftrightarrow \mathrm{Re}[\langle x, y\rangle] < 0$;

(*j*) $\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$; (*Parallelogram law*)

(*k*) *if* $\|x\| = \|y\| = 1$, *then* $\|x+y\|^2 + \|x-y\|^2 = 4$. (*Rhombus identity*)

Corollary 2.4 NORM EXPANSIONS OF INNER PRODUCTS. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Then the following properties hold for all $x, y \in H$:*

(*a*) $\mathrm{Re}[\langle x, y\rangle] = \frac{1}{2}(\langle x, y\rangle + \langle y, x\rangle)$;

(*b*) $\mathrm{Im}[\langle x, y\rangle] = \frac{i}{2}(\langle y, x\rangle - \langle x, y\rangle) = \mathrm{Re}[\langle x, iy\rangle] = \frac{1}{2}(\langle x, iy\rangle + \langle iy, x\rangle)$;

(*c*) $\langle x, y\rangle + \langle y, x\rangle = \|x+y\|^2 - \|x\|^2 + \|y\|^2 = \|x\|^2 + \|y\|^2 - \|x-y\|^2$;

(*d*) $\mathrm{Re}[\langle x, y\rangle] = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2)$;

(e) *if* $\mathbb{F} = \mathbb{R}$,
$\langle x, y\rangle = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2)$
and
$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow \langle x, y\rangle = 0$.

PROOF. Write $\langle x, y\rangle = z_1 + iz_2 \in \mathbb{C}$ where $z_1, z_2 \in \mathbb{R}$. Then

$$\begin{aligned}\langle x, y\rangle + \langle y, x\rangle &= \langle x, y\rangle + \langle y, x\rangle = \langle x, y\rangle + \overline{\langle x, y\rangle} \\ &= (z_1 + iz_2) + (z_1 - iz_2) = 2z_1 = 2\,\mathrm{Re}[\langle x, y\rangle]\,,\end{aligned} \tag{2.22}$$

which yields (a). Similarly,

$$\begin{aligned}\langle y, x\rangle - \langle x, y\rangle &= \overline{\langle x, y\rangle} - \langle x, y\rangle = (z_1 - iz_2) - (z_1 + iz_2) \\ &= -i(2z_2) = -i(2\,\mathrm{Im}[\langle x, y\rangle])\end{aligned} \tag{2.23}$$

so that

$$\mathrm{Im}[\langle x, y\rangle] = \frac{i}{2}(\langle y, x\rangle - \langle x, y\rangle)\,; \tag{2.24}$$

further,

$$\langle x, iy\rangle = -i\,\langle x, y\rangle = -i(z_1 + iz_2) = z_2 - iz_1 \tag{2.25}$$

so that

$$\mathrm{Im}[\langle x, y\rangle] = z_2 = \mathrm{Re}[\langle x, iy\rangle] = \frac{1}{2}(\langle x, iy\rangle + \langle iy, x\rangle)\,. \tag{2.26}$$

This establishes (b). By Proposition 2.3 (d), we have:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle\;, \tag{2.27}$$

$$\begin{aligned}\|x-y\|^2 &= \|x\|^2 + \|-y\|^2 + \langle x, -y\rangle + \langle -y, x\rangle = \|x\|^2 + \|y\|^2 + \overline{\langle -y, x\rangle} + \langle -y, x\rangle \\ &= \|x\|^2 + \|y\|^2 - \overline{\langle y, x\rangle} - \langle y, x\rangle = \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle\;.\end{aligned} \tag{2.28}$$

Solving for $\langle x, y\rangle + \langle y, x\rangle$, these yield (c). (d) is entailed by (a) and (c). (e) follows from (c) on observing that $\langle x, y\rangle = \langle y, x\rangle$ and $\langle x, y\rangle = \mathrm{Re}[\langle x, y\rangle]$ when $\mathbb{F} = \mathbb{R}$. □

Proposition 2.5 BEST APPROXIMATION OF A VECTOR BY A LINE. *Let H be an inner-product space, $x, y \in H$, and*

$$\lambda_0 := \frac{\langle y, x\rangle}{\|x\|^2} = \bar{\beta}_H(x, y)\,, \quad \rho_H(x, y) := \frac{\langle x, y\rangle}{\|y\|\,\|x\|}\,. \tag{2.29}$$

Then,

$$\langle x, y - \lambda_0 x\rangle = \langle y - \lambda_0 x, x\rangle = 0\,, \tag{2.30}$$

$$\begin{aligned} \|y-\lambda_0 x\|^2 &= \|y\|^2\left(1-|\rho_H(x,y)|^2\right) \\ &= \begin{cases} \left(\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right)/\|x\|^2 & \mathit{if}\ \|x\|\neq 0 \\ \|y\|^2 & \mathit{if}\|x\|=0 \end{cases}, \end{aligned} \tag{2.31}$$

$$\|y-\lambda_0 x\|=0 \Rightarrow |\langle x,y\rangle| = \|x\|\,\|y\|\,, \tag{2.32}$$

$$\|y-\lambda_0 x\|=0 \Leftrightarrow \{[\|y\|=0]\ \mathit{or}\ [\|y\|\neq 0\ \mathit{and}\ \ |\langle x,y\rangle| = \|x\|\,\|y\|]\}\,, \tag{2.33}$$

and, for any $\lambda \in \mathbb{F}$,

$$\langle \lambda x, y-\lambda_0 x\rangle = \langle y-\lambda_0 x, \lambda x\rangle = 0\,, \tag{2.34}$$

$$\|y-\lambda x\|^2 = \|y-\lambda_0 x\|^2 + |\lambda-\lambda_0|^2\|x\|^2 \geq \|y-\lambda_0 x\|^2 \tag{2.35}$$

with

$$\|y-\lambda x\|^2 > \|y-\lambda_0 x\|^2 \Leftrightarrow [\lambda \neq \lambda_0\ \mathit{and}\ \|x\|\neq 0]\,. \tag{2.36}$$

PROOF OF PROPOSITION 2.5 We have

$$\begin{aligned} \langle x, y-\lambda_0 x\rangle &= \langle x,y\rangle - \langle x, \lambda_0 x\rangle = \langle x,y\rangle - \bar{\lambda}_0\langle x,x\rangle = \langle x,y\rangle - \frac{\overline{\langle y,x\rangle}}{\|x\|^2}\langle x,x\rangle \\ &= \langle x,y\rangle - \frac{\langle x,y\rangle}{\|x\|^2}\langle x,x\rangle = \langle x,y\rangle - \langle x,y\rangle = 0\,, \end{aligned} \tag{2.37}$$

$$\langle y-\lambda_0 x, x\rangle = \overline{\langle x, y-\lambda_0 x\rangle} = 0\,, \tag{2.38}$$

hence

$$\begin{aligned} \|y-\lambda_0 x\|^2 &= \langle y-\lambda_0 x, y-\lambda_0 x\rangle \\ &= \langle y, y-\lambda_0 x\rangle - \lambda_0\langle x, y-\lambda_0 x\rangle = \langle y, y-\lambda_0 x\rangle \\ &= \langle y,y\rangle - \bar{\lambda}_0\langle y,x\rangle = \|y\|^2 - \frac{\overline{\langle y,x\rangle}}{\|x\|^2}\langle y,x\rangle \\ &= \|y\|^2 - \frac{|\langle y,x\rangle|^2}{\|x\|^2} = \|y\|^2\left(1-\frac{|\langle y,x\rangle|^2}{\|y\|^2\|x\|^2}\right) \\ &= \|y\|^2\left(1-|\rho_H(y,x)|^2\right) \end{aligned} \tag{2.39}$$

where we use the conventions $|\langle y,x\rangle|^2/\|x\|^2 = 0$ and $\rho_H(y,x)=0$ when $\|x\|=0$ and/or $\|y\|=0$. If $\|x\|=0$, we thus have $\|y-\lambda_0 x\|^2 = \|y\|^2$, while for $\|x\|\neq 0$,

$$\|y-\lambda_0 x\|^2 = \|y\|^2 - \frac{|\langle y,x\rangle|^2}{\|x\|^2} = \frac{\|x\|^2\|y\|^2 - |\langle y,x\rangle|^2}{\|x\|^2}\,. \tag{2.40}$$

This establishes (2.31).

Now, suppose that $\|y-\lambda_0 x\|=0$. This can hold with either $\|x\|=0$ or $\|x\|\neq 0$. If $\|x\|=0$ we have $\langle y,x\rangle = 0$ and $\lambda_0 = 0$, hence $\|y-\lambda_0 x\| = \|y\| = 0$, $\|y\|=0$ and $|\langle y,x\rangle| = \|x\|\,\|y\|$. If $\|x\|\neq 0$, then

(2.31) implies again that $|\langle y, x\rangle| = \|x\|\,\|y\|$. This shows (2.32).
To get the equivalence (2.33), we first note that

$$\begin{aligned}|\langle y, x\rangle| = \|x\|\,\|y\| \quad &\Rightarrow \quad \{[\|y\| = 0 \text{ and } |\langle y, x\rangle| = \|x\|\,\|y\|] \text{ or } [y \neq \mathbf{0} \text{ and } |\langle y, x\rangle| = \|x\|\,\|y\|]\} \\ &\Rightarrow \quad \{[\|y\| = 0] \text{ or } [\|y\| \neq 0 \text{ and } |\langle y, x\rangle| = \|x\|\,\|y\|]\}\,. \end{aligned} \tag{2.41}$$

Conversely, each one the two conditions $[\|y\| = 0]$ and $[\|y\| \neq 0 \text{ and } |\langle y, x\rangle| = \|x\|\,\|y\|]$ implies $|\langle y, x\rangle| = \|x\|\,\|y\|$, hence

$$\{[\|y\| = 0] \text{ or } [\|y\| \neq 0 \text{ and } |\langle y, x\rangle| = \|x\|\,\|y\|]\} \Rightarrow |\langle y, x\rangle| = \|x\|\,\|y\|\ . \tag{2.42}$$

This establishes (2.33).
From (2.37), we get:

$$\langle \lambda x,\, y - \lambda_0 x\rangle = \lambda\, \langle x,\, y - \lambda_0 x\rangle = 0 = \bar{\lambda}\, \langle y - \lambda_0 x,\, x\rangle = \langle y - \lambda_0 x,\, \lambda x\rangle \tag{2.43}$$

for any $\lambda \in \mathbb{F}$, and thus (2.34). Further, we can write

$$y - \lambda x = (y - \lambda_0 x) + (\lambda - \lambda_0)x\,, \quad \text{for } \lambda \in \mathbb{F}, \tag{2.44}$$

and, since $\langle y - \lambda_0 x,\, (\lambda - \lambda_0)x\rangle = 0$,

$$\begin{aligned}\|y - \lambda x\|^2 \quad &= \quad \|y - \lambda_0 x\|^2 + \|(\lambda - \lambda_0)x\|^2 \\ &= \quad \|y - \lambda_0 x\|^2 + |\lambda - \lambda_0|^2\,\|x\|^2 \geq \|y - \lambda_0 x\|^2 \end{aligned} \tag{2.45}$$

where the inequality is strict when $|\lambda - \lambda_0|^2\,\|x\|^2 > 0$, *i.e.* when $\|x\| > 0$ and $\lambda \neq \lambda_0$. This completes the proof. $\square$

Proposition 2.6 CAUCHY-SCHWARZ INEQUALITY. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner-product space. Then, for all $x, y \in H$,*

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \tag{2.46}$$

and

$$\begin{aligned}|\langle x, y\rangle| = \|x\|\,\|y\| \quad &\Leftrightarrow [(x = \mathbf{0}) \textit{ or } (y = \lambda x \textit{ for some } \lambda \in \mathbb{F})] \\ &\Leftrightarrow [(x \neq \mathbf{0}) \Rightarrow (y = \lambda x \textit{ for some } \lambda \in \mathbb{F})] \\ &\Leftrightarrow [(x = \lambda y \textit{ for some } \lambda \in \mathbb{F}) \textit{ or } (y = \lambda x \textit{ for some } \lambda \in \mathbb{F})] \\ &\Leftrightarrow [(x \neq \lambda y \textit{ for all } \lambda \in \mathbb{F}) \textit{ or } (y = \lambda x \textit{ for some } \lambda \in \mathbb{F})]\,. \end{aligned} \tag{2.47}$$

If $\mathbb{F} = \mathbb{R}$, then, for all $x, y \in H$,

$$\begin{aligned}\langle x, y\rangle = \|x\|\,\|y\| \quad &\Leftrightarrow [(x = \mathbf{0}) \textit{ or } (y = \lambda x \textit{ for some nonnegative } \lambda \in \mathbb{R})] \\ &\Leftrightarrow [(x \neq \mathbf{0}) \Rightarrow (y = \lambda x \textit{ for some nonnegative } \lambda \in \mathbb{F})] \\ &\Leftrightarrow [(x = \lambda y \textit{ for some } \lambda \in \mathbb{F}) \textit{ or } (y = \lambda x \textit{ for some } \lambda \in \mathbb{F})] \\ &\Leftrightarrow [(x \neq \lambda y \textit{ for all } \lambda \in \mathbb{F}) \Rightarrow (y = \lambda x \textit{ for some } \lambda \in \mathbb{F})]\,. \end{aligned} \tag{2.48}$$

Proposition 2.7 TRIANGLE INEQUALITY. *Let $(H, \langle\cdot,\,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x,\,x\rangle}$. Then the following properties hold for all $x, y \in H$:*

(a) $\|x+y\| \leq \|x\| + \|y\|$; (*Triangle inequality*)

(b) $|\,\|x\| - \|y\|\,| \leq \|x+y\|$;

(c) $\|x+y\| = \|x\| + \|y\| \Leftrightarrow \mathrm{Re}[\langle x,\,y\rangle] = \|x\|\,\|y\|$;

(d) $\|x+y\| < \|x\| + \|y\| \Leftrightarrow \mathrm{Re}[\langle x,\,y\rangle] \neq \|x\|\,\|y\|$. (*Strict triangle inequality*)

3. Special inner-product identities and inequalities

Proposition 3.1 POLARIZATION IDENTITIES. *Let $(H, \langle\cdot,\,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x,\,x\rangle}$. Then the following properties hold for all $x, y \in H$:*

(a) $\langle x,\,y\rangle + \langle y,\,x\rangle = \frac{1}{2}(\|x+y\|^2 - \|x-y\|^2)$; (*General polarization identity*)

(b) $\mathrm{Re}[\langle x,\,y\rangle] = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$; (*Real polarization identity*)

(c) $\mathrm{Im}[\langle x,\,y\rangle] = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$; (*Imaginary polarization identity*)

(d) $\langle x,\,y\rangle = \frac{1}{4}\left[(\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2)\right]$; (*Complex polarization identity*)

(e) *if* $\mathbb{F} = \mathbb{R}$,
$\langle x,\,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$. (*Polarization identity for real vector spaces*)

PROOF. By Corollary 2.4, we have:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x,\,y\rangle + \langle y,\,x\rangle\ , \tag{3.1}$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - [\langle x,\,y\rangle + \langle y,\,x\rangle]\,, \tag{3.2}$$

hence

$$\|x+y\|^2 - \|x-y\|^2 = 2[\langle x,\,y\rangle + \langle y,\,x\rangle] = 2[\langle x,\,y\rangle + \overline{\langle x,\,y\rangle}] = 4\,\mathrm{Re}[\langle x,\,y\rangle] \tag{3.3}$$

and

$$\langle x,\,y\rangle + \langle y,\,x\rangle = \frac{1}{2}(\|x+y\|^2 - \|x-y\|^2)\,, \tag{3.4}$$

$$\mathrm{Re}[\langle x,\,y\rangle] = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\}\,. \tag{3.5}$$

This yields both the general polarization identity and the real polarization identity. Using (3.3), we also get:

$$\begin{aligned}\|x+iy\|^2 - \|x-iy\|^2 &= 2[\langle x,\,iy\rangle + \langle iy,\,x\rangle] = 2[\overline{\langle iy,\,x\rangle} + i\,\langle y,\,x\rangle] \\ &= 2[\overline{(i)}\,\overline{\langle y,\,x\rangle} + i\,\langle y,\,x\rangle] = 2[(-i)\,\overline{\langle y,\,x\rangle} + i\,\langle y,\,x\rangle] = 2[-i\,\langle x,\,y\rangle + i\,\langle y,\,x\rangle]\end{aligned}$$

$$
\begin{aligned}
&= -2i[\langle x, y\rangle - \langle y, x\rangle] = -2i[\langle x, y\rangle - \overline{\langle x, y\rangle}] && (3.6)\\
&= -2i\{(\mathrm{Re}[\langle x, y\rangle] + i\,\mathrm{Im}[\langle x, y\rangle]) - (\mathrm{Re}[\langle x, y\rangle] - i\,\mathrm{Im}[\langle x, y\rangle])\} && (3.7)\\
&= -2i\{(2i)\,\mathrm{Im}[\langle x, y\rangle] = 4\,\mathrm{Im}[\langle x, y\rangle] && (3.8)
\end{aligned}
$$

hence

$$
\mathrm{Im}[\langle x, y\rangle] = -\frac{1}{4}(\|x-iy\|^2 - \|x+iy\|^2) = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2) \tag{3.9}
$$

which establishes the imaginary polarization identity. The complex polarization identity is a direct consequence of the two previous ones. The polarization identity for real vector spaces follows from observing that $\langle x, y\rangle = \langle y, x\rangle$ when $\mathbb{F} = \mathbb{R}$. □

Proposition 3.2 APOLLONIUS' IDENTITY. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Then, for all $x_1, x_2, y \in H$:*

$$
\|y-x_1\|^2 + \|y-x_2\|^2 = 2\,\|y - [(x_1+x_2)/2]\|^2 + \frac{1}{2}\,\|x_1-x_2\|^2\,, \tag{3.10}
$$

$$
\|y - [(x_1+x_2)/2]\|^2 = \frac{1}{2}(\|y-x_1\|^2 + \|y-x_2\|^2) - \frac{1}{4}\,\|x_1-x_2\|^2\,, \tag{3.11}
$$

$$
\|y - [(x_1+x_2)/2]\|^2 \le \frac{1}{2}(\|y-x_1\|^2 + \|y-x_2\|^2)\,, \tag{3.12}
$$

$$
\|x_1-x_2\|^2 \le 2(\|y-x_1\|^2 + \|y-x_2\|^2)\,. \tag{3.13}
$$

PROOF. Using the parallelogram law, we first observe that

$$
\begin{aligned}
\|y-x_1\|^2 + \|y-x_2\|^2 &= \frac{1}{2}(\|(y-x_1)+(y-x_2)\|^2 + \|(y-x_1)-(y-x_2)\|^2) && (3.14)\\
&= \frac{1}{2}(\|(y-x_1)+(y-x_2)\|^2 + \|x_1-x_2\|^2) && (3.15)
\end{aligned}
$$

hence

$$
\|(y-x_1)+(y-x_2)\|^2 = 2(\|y-x_1\|^2 + \|y-x_2\|^2) - \|x_1-x_2\|^2\,, \tag{3.16}
$$

$$
\begin{aligned}
\|y - [(x_1+x_2)/2]\|^2 &= \frac{1}{4}\,\|(y-x_1)+(y-x_2)\|^2 && (3.17)\\
&= \frac{1}{2}[\|y-x_1\|^2 + \|y-x_2\|^2] - \frac{1}{4}\,\|x_1-x_2\|^2]\,, && (3.18)
\end{aligned}
$$

and

$$
\frac{1}{2}[\|y-x_1\|^2 + \|y-x_2\|^2] = \|y - [(x_1+x_2)/2]\|^2 + \frac{1}{4}\,\|x_1-x_2\|^2]\,, \tag{3.19}
$$

$$
\|y-x_1\|^2 + \|y-x_2\|^2 = 2\,\|y - [(x_1+x_2)/2\|^2 + \frac{1}{2}\,\|x_1-x_2\|^2\,. \tag{3.20}
$$

This establishes the first identity. The other identity and inequalities are straightforward implications of the latter. $\square$

Proposition 3.3 SUMS OF INNER PRODUCTS. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner-product space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Then, for all $x_1, y_1, x_2, y_2 \in H$,*

$$\langle x_1, y_2\rangle + \langle x_2, y_1\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle - \langle x_1 - x_2, y_1 - y_2\rangle\,, \tag{3.21}$$

$$\langle x_1, x_2\rangle + \langle x_2, x_1\rangle = 2\,\mathrm{Re}[\langle x_1, x_2\rangle] = \|x_1\|^2 + \|x_2\|^2 - \|x_1 - x_2\|^2\,. \tag{3.22}$$

PROOF. The first identity follows on noting that

$$\langle x_1 - x_2, y_1 - y_2\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle - \langle x_1, y_2\rangle - \langle x_2, y_1\rangle \tag{3.23}$$

and then solving for $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle$. The second identity follows on taking $y_1 = x_1$ and $y_2 = x_2$ in (3.21), and using the identity [see Corollary 2.4 (a)]

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = 2\,\mathrm{Re}[\langle x_1, x_2\rangle]\,. \tag{3.24}$$

$\square$

Proposition 3.4 PTOLEMAIC IDENTITIES AND INEQUALITIES. *Let $(H, \langle\cdot,\cdot\rangle)$ be a real inner-product space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Then for all $x_1, y_1, x_2, y_2 \in H$:*

(a)

$$\langle x_1 - x_2, y_1 - y_2\rangle = \langle x_1 - y_1, x_2 - y_2\rangle + \langle x_1 - y_2, y_1 - x_2\rangle\,; \tag{3.25}$$

(b)

$$\begin{aligned} |\langle x_1 - x_2, y_1 - y_2\rangle| &\leq |\langle x_1 - y_1, x_2 - y_2\rangle| + |\langle x_1 - y_2, y_1 - x_2\rangle| \\ &\leq \|x_1 - y_1\|\,\|x_2 - y_2\| + \|x_1 - y_2\|\,\|x_1 - y_2\|\,; \end{aligned} \tag{3.26}$$

(c) *if the sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are orthogonal,*

$$\langle x_1 - x_2, y_1 - y_2\rangle = \langle x_1 - y_2, y_1 - x_2\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle\,, \tag{3.27}$$

$$|\langle x_1 - x_2, y_1 - y_2\rangle| = |\langle x_1 - y_2, y_1 - x_2\rangle| \leq |\langle x_1, y_1\rangle| + |\langle x_2, y_2\rangle|\,; \tag{3.28}$$

(d) *if the sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are orthogonal and $\|x_1\| = \|x_2\| = \|y_1\| = \|y_2\| = 1$,*

$$\rho_H(x_1 - x_2, y_1 - y_2) = \frac{1}{2}[\rho_H(x_1, y_1) + \rho_H(x_2, y_2)]\,. \tag{3.29}$$

PROOF. To show (3.25), we note that:

$$\langle x_1 - y_1, x_2 - y_2\rangle = \langle x_1, x_2\rangle - \langle x_1, y_2\rangle - \langle y_1, x_2\rangle + \langle y_1, y_2\rangle\,, \tag{3.30}$$

$$\langle x_1 - y_2, y_1 - x_2\rangle = \langle x_1, y_1\rangle - \langle x_1, x_2\rangle - \langle y_2, y_1\rangle + \langle y_2, x_2\rangle \,, \tag{3.31}$$

hence, taking the sum,

$$\begin{aligned}\langle x_1 - y_1, x_2 - y_2\rangle + \langle x_1 - y_2, y_1 - x_2\rangle &= \langle x_1, y_1\rangle + \langle y_2, x_2\rangle - \langle x_1, y_2\rangle - \langle y_1, x_2\rangle \\ &= \langle x_1 - x_2, y_1 - y_2\rangle \,. \end{aligned} \tag{3.32}$$

The following identities and inequalities are simple implications of (3.25). For the last one, we note that

$$\|x_1 - x_2\|^2 = \|y_1 - y_2\|^2 = 2 \tag{3.33}$$

when $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are orthogonal and $\|x_1\| = \|x_2\| = \|y_1\| = \|y_2\| = 1$, hence

$$\begin{aligned}\rho_H(x_1 - x_2, y_1 - y_2) &= \frac{\langle x_1 - x_2, y_1 - y_2\rangle}{\|x_1 - x_2\|\,\|y_1 - y_2\|} = \frac{\langle x_1 - x_2, y_1 - y_2\rangle}{2} \\ &= \frac{1}{2}[\langle x_1, y_1\rangle + \langle x_2, y_2\rangle] = \frac{1}{2}[\rho_H(x_1, y_1) + \rho_H(x_2, y_2)] \,. \end{aligned} \tag{3.34}$$

□

4. Normed and metric spaces

Definition 4.1 NORMED SPACE. *Let B be a vector space on a scalar set $\mathbb{F}$. A* norm *on H is an application which associates to each element x of H a scalar $\|x\| \in \mathbb{R}$ such that, for all $x,\ y,\ z \in H$,*

(a) $\|x\| \geq 0$;

(b) $\|x\| = 0$ *if and only if* $x = 0$;

(c) $\|x + y\| \leq \|x\| + \|y\|$; (*Triangle inequality*)

(d) $\|\alpha x\| = |\alpha|\,\|x\|$ *for all* $\alpha \in \mathbb{F}$.

If $\|\cdot\|$ is a norm on B, the pair $(B, \|\cdot\|)$ is called an normed space*, and the elements of B are also called elements of the normed space $(B, \|\cdot\|)$. If $\mathbb{F} = \mathbb{R}$, we say that $(B, \|\cdot\|)$ is a* real normed space*, and if $\mathbb{F} = \mathbb{C}$, we say that $(B, \langle\cdot, \cdot\rangle)$ is a* complex normed space*. When there is no ambiguity on the definition of the inner product, the normed space $(B, \|\cdot\|)$ may simply be denoted B.*

In inner-product spaces, it is easy to see that $\|x\| := \sqrt{\langle x, x\rangle}$ satisfies all the properties listed in Definition 4.1.

Definition 4.2 METRIC SPACE. *Let X be nonempty space of elements* (*called* points)*. A metric d on X is a real-valued function on $X \times X$ such that, for all x, y and z in X:*

(a) $d(x, y) \geq 0$;

(b) $\quad d(x,y) = 0$ *if and only if* $x = y$;

(c) $\quad d(x,y) = d(y,x)$;

(d) $\quad d(x,y) \leq d(x,z) + d(z,y)$.

The pair (X,d) *is called a metric space.*

In normed spaces, it is easy to see that $d(x,y) := \|x - y\|$ satisfies all the properties listed in Definition 4.2. Thus, inner-product spaces are a subclass of normed spaces, which are themselves a subclass of metric spaces. Properties established for metric spaces also hold for normed spaces, and properties established on normed spaces apply in inner-product spaces.

5. Interior, closure and boundary of a set

The notion of metric space allows one to define open and closed sets.

Definition 5.1 INTERIOR POINT. *Let S be a subset of an inner-product space H. A point* $y \in S$ *is an* interior point *of S if and only if there exists* $\varepsilon > 0$ *such that* $\{x : \|x - y\| < \varepsilon\} \subseteq S$. *The set of all the interior points of S is called the interior of S and is denoted* $\mathrm{int}(S)$.

Definition 5.2 OPEN SET. *Let S be a subset of an inner-product space H. S is an* open *set if and only if* $S = \mathrm{int}(S)$.

Definition 5.3 CLOSURE POINT. *Let S be a subset of an inner-product space H. A point* $y \in H$ *is a* closure point *of S if and only if for any* $\varepsilon > 0$, *the set* $\{x : \|x - y\| < \varepsilon\}$ *contains at least one point in* $x \in S$. *The set of all the closure points of S is called the* closure *of S and is denoted* $\mathrm{cl}(S)$.

Definition 5.4 CLOSED SET. *Let S be a subset of an inner-product space H. S is a* closed *set if and only if* $S = \mathrm{cl}(S)$.

Definition 5.5 BOUNDARY OF A SET. *Let S be a subset of an inner-product space H. The boundary of S is* $\partial S := \mathrm{cl}(S) \backslash \mathrm{int}(S)$. *Any element of* ∂S *is called a* boundary points *of S.*

Definition 5.6 ISOLATED POINT. *Let S be a subset of an inner-product space H. A point* $y \in S$ *is an* isolated point *of S if and only if there exists* $\varepsilon > 0$ *such that the set* $\{x : \|x - y\| < \varepsilon\}$ *does not contain any point of S except y.*

Proposition 5.1 *Let S be a subset of an inner-product space H. Then the following properties hold:*

(a) $\quad \mathrm{int}(S) \subseteq S \subseteq \mathrm{cl}(S)$ *;*

(b) $\quad \partial S \subseteq \mathrm{cl}(S)$ *and* $\partial S \cap \mathrm{int}(S) = \emptyset$ *;*

(c) $\quad$ *if y is an isolated point of S, then* $y \in \partial S$*;*

(d) $\quad \mathrm{int}(\mathrm{int}(S)) = \mathrm{int}(S)$ *;*

(e) $\quad \mathrm{cl}(\mathrm{cl}(S)) = \mathrm{cl}(S)$ *.*

PROOF. The proposition follows directly from applying the definitions $\mathrm{int}(S)$, $\mathrm{cl}(S)$ and ∂S. □

Proposition 5.2 *Let H be a subset of an inner-product space. Then the following properties hold:*

(a) *the complement of an open set is closed;*

(b) *the complement of a closed set is open;*

(c) *the intersection of a finite number of open sets is open;*

(d) *the union of an arbitrary collection of open sets is open;*

(e) *the union of a finite number of closed sets is closed;*

(f) *the intersection of an arbitrary collection of closed sets is closed.*

PROOF. See Luenberger (1969, Section 2.7). □

Proposition 5.3 *Let S be a convex subset of an inner-product space H. Then* $\mathrm{int}(S)$ *and* $\mathrm{cl}(S)$ *are convex.*

PROOF. See Luenberger (1969, Section 2.7). □

6. Convergence

Definition 6.1 CONVERGENCE IN NORM. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of a normed space* $(B, \|\cdot\|)$*. We say that* x_n *converges to* $x \in B$ *if and only if*

$$\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0\,. \tag{6.1}$$

Convergence in norm in B is denoted $x_n \underset{n\to\infty}{\overset{B}{\longrightarrow}} x$ *or, when the context is clear,* $x_n \underset{n\to\infty}{\longrightarrow} x$*.*

Definition 6.2 CLOSURE. *Let S be a nonempty subset of an inner-product space H. Then the* closure *of S, denoted* $\bar{S}$*, is the smallest closed set which contains S.*

Proposition 6.1 CONTINUITY OF THE NORM. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of an normed space* $(B, \langle\cdot,\cdot\rangle)$ *such that* $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0$ *where* $x \in B$. *Then*

$$\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\| . \tag{6.2}$$

Definition 6.3 CAUCHY SEQUENCE. *Let* S *be a nonempty subset of an inner-product space* $(H, \langle\cdot,\cdot\rangle)$, *and* $\{x_n : n = 1, 2, ...\} \subseteq S$ *a sequence of elements in* S. *Then* x_n *is a* Cauchy sequence *in* S *if and only if*

$$\|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0 . \tag{6.3}$$

Proposition 6.2 RELATION BETWEEN CONVERGENCE TO A POINT AND CAUCHY CONVERGENCE. *Let* $\{x_n : n = 1, 2, ...\}$ *be a sequence of elements of an inner-product space* $(H, \langle\cdot,\cdot\rangle)$ *and* $x \in H$. *If* $x_n \underset{n\to\infty}{\overset{H}{\longrightarrow}} x$, *then* x_n *is a* Cauchy sequence *in* S.

Definition 6.4 COMPLETE SUBSET. *Let* $(H, \langle\cdot,\cdot\rangle)$ *an inner-product space, and* S *a subset of* H. *The subset* S *is* complete *if and only if every Cauchy sequence in* S *converges to a point in* S*:* i.e.,

$$[\{x_n : n = 1,\, 2,\, ...\} \subseteq S \text{ and } \|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0] \;\Rightarrow\; \exists x \in S \text{ such that } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 . \tag{6.4}$$

Proposition 6.3 CONVERGENCE CHARACTERIZATION OF CLOSED SUBSETS. *Let* $(H, \langle\cdot,\cdot\rangle)$ *an inner-product space, and* S *a subset of* H. S *is* closed *if and only if* S *contains all its limit points,* i.e.

$$x_n \in S\,,\ \forall n\,,\ \text{and } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \;\Rightarrow\; x \in S . \tag{6.5}$$

Proposition 6.4 RELATION BETWEEN COMPLETE AND CLOSED SUBSETS. *Let* $(H, \langle\cdot,\cdot\rangle)$ *an inner-product space, and* S *a subset of* H. *If* S *is complete, then* S *is closed. If* H *is complete, then*

$$S \text{ is complete } \Leftrightarrow S \text{ is closed.} \tag{6.6}$$

S is complete, then S is closed.

Proposition 6.5 CONTINUITY OF INNER PRODUCT. *Let* $\{x_n : n = 1, 2, ...\}$ *and* $\{y_n : n = 1, 2, ...\}$ *be two sequences of elements of an inner-product space* $(H, \langle\cdot,\cdot\rangle)$ *such that* $\|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0$ *and* $\|y_n - y\| \underset{n\to\infty}{\longrightarrow} 0$, *where* $x, y \in H$. *Then*

$$\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\| \tag{6.7}$$

and

$$\langle x_n, y_n\rangle \underset{n\to\infty}{\longrightarrow} \langle x, y\rangle . \tag{6.8}$$

Example 6.1 MEAN SQUARE CONVERGENCE. Let $\{X_n : n = 1, 2, \ldots\}$ a sequence of real random variables in $L^2 \equiv L^2(\Omega, \mathscr{A}, P)$. If we define $\langle X, Y \rangle = E(XY)$, then the convergence in norm represents mean-square convergence:

$$\|X_n - X\| \underset{n\to\infty}{\longrightarrow} 0 \Leftrightarrow E[(X_n - X)^2] \underset{n\to\infty}{\longrightarrow} 0\,. \tag{6.9}$$

7. Hilbert spaces

Definition 7.1 HILBERT SPACE. *Let $(H, \langle\cdot,\cdot\rangle)$ an inner-product space. Then $(H, \langle\cdot,\cdot\rangle)$ is a* Hilbert space *if and only if every Cauchy sequence of $(H, \langle\cdot,\cdot\rangle)$ converges in norm to an element of H,*

$$\|x_n - x_m\| \underset{m,n\to\infty}{\longrightarrow} 0 \Rightarrow \exists x \in H \text{ such that } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0\,. \tag{7.1}$$

Proposition 7.1 NORM CONVERGENCE AND CAUCHY CRITERION. *Let $\{x_n : n = 1, 2, \ldots\}$ be a sequence of elements of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then x_n converges in norm if and only if x_n is a Cauchy sequence.*

Theorem 7.2 COMPLETENESS OF $L^2(\Omega, \mathscr{A}, P)$. *The inner-product space $L^2(\Omega, \mathscr{A}, P)$ with $\langle X, Y \rangle = E(XY)$ is complete and thus constitutes a Hilbert space.*

PROOF. See Brockwell and Davis (1991, Section 2.10). □

Definition 7.2 CLOSED SUBSPACE. *A subspace S of a Hilbert space H is a* closed subspace *of H if and only if S contains all its limit points,* i.e.

$$x_n \in S,\ \forall n,\ \text{and } \|x_n - x\| \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow x \in S\,. \tag{7.2}$$

8. Sources and additional references

For good reviews of vector space theory, see Halmos (1958), Herstein (1975), Kolmogorov and Fomin (1975, Chapters 4-6). A good summary of Hilbert space theory aimed at applications in time series analysis may be found in Brockwell and Davis (1991, Chapter 2). Other good reviews appear in: Debnath and Mikusiński (1990) for general applications, Small and McLeish (1994) for applications in statistical theory, Luenberger (1969), Deutsch (2001), Aubin (2007), Bauschke and Combettes (2011) for applications to optimization problems, and Young (1988) for a more mathematically oriented presentation.

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