Unit root tests *

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1. Unit root tests in AR(1) models

1.1. Random walk without drift

Consider the model:

$$Y_t = \varphi\, Y_{t-1} + u_t,\ t = 1, 2, \ldots, T, \tag{1.1}$$

$$Y_0 = 0, \tag{1.2}$$

$$(u_t : t = 1, \ldots, T) \sim IID\left(0, \sigma^2\right) \tag{1.3}$$

We wish to test the hypothesis:

$$H_0 : \varphi = 1 \tag{1.4}$$

Let $\hat{\varphi}$ be the estimator of φ obtained by ordinary least squares (OLS) from equation (1.1). In contrast with what happens when $|\varphi| < 1$, the asymptotic distribution of $\hat{\varphi}$ is not normal. More precisely, we can show that

$$\underset{T \longrightarrow \infty}{p \lim} \sqrt{T}\,(\hat{\varphi} - 1) = 0.$$

However, $T\,(\hat{\varphi} - 1)$ has an asymptotic distribution which does not depend on σ^2. Similarly, the t statistic associated with H_0,

$$t_{\varphi-1} = (\hat{\varphi} - 1) \,/\, \left[s^2 \left(\sum_{t=1}^{T} Y_{t-1}^2 \right)^{-1} \right]^{1/2} .$$

has an asymptotic distribution which does not depend on σ^2. However, the asymptotic distributions of $T\,(\hat{\varphi} - 1)$ and $t_{\varphi-1}$ are not normal.

Usually the null hypothesis $H_0 : \varphi = 1$ is tested against an alternative of stationarity ($\varphi < 1$), so that it is natural to consider one-sided critical regions of the form:

$$T(\hat{\varphi} - 1) < c_1(\alpha) \tag{1.5}$$

or

$$t_{\varphi-1} < c_2(\alpha) \tag{1.6}$$

where α is the level of the test.

For the case where the u_t's are normal, the exact critical points for $T\,(\hat{\varphi} - 1)$ and $t_1\,(\varphi = 1)$ are given by Fuller (1976, pp. 373 and 375): Table 8.5.1 $(\hat{\rho})$ for $T\,(\hat{\varphi} - 1)$ and Table 8.5.2 $(\hat{\tau})$ for $t_{\varphi-1}$. These critical points are also asymptotically valid (with $T = \infty$) in the case where the u_t's are not normal.

If $Y_0 \neq 0$, the same tests can also be applied after replacing Y_t by $Y_t - Y_0$, $i.e.$ we consider the regression;

$$Y_t - Y_0 = \varphi (Y_{t-1} - Y_0) + u_t,\ t = 1, \ldots, T. \tag{1.7}$$

1.2. Random walk with drift

Model :

$$Y_t = \mu_0 + \varphi Y_{t-1} + u_t,\ t = 1, \ldots, T, \tag{1.8}$$

$$Y_0 \text{ is fixed,} \tag{1.9}$$

$$(u_t : t = 1, \ldots, T) \sim IID\left(0, \sigma^2\right). \tag{1.10}$$

Hypothesis:

$$H_0 : \varphi = 1. \tag{1.11}$$

Let $\hat{\varphi}$ be the OLS estimator of φ based on equation (1.8) and $t_{\varphi-1}$ the t statistic associated with H_0. For testing H_0, we can use either $T(\hat{\varphi} - 1)$ or $t_{\varphi-1}$. For the case where the u_t's are normal, exact critical points are given by Fuller (1976, pp. 371 and 373): Table 8.5.1 $\left(\hat{\rho}_\mu\right)$ for $T(\hat{\varphi} - 1)$ and Table 8.5.2 $(\hat{\tau}_\mu)$ for $t_{\varphi-1}$. These critical points are also asymptotically valid (with $T = \infty$) when the u_t's are not normal.

In (1.8), we may also wish to test the null hypothesis

$$H_{01} : \varphi = 1 \text{ and } \mu_0 = 0\,. \tag{1.12}$$

This can be done by computing the usual $F-$statistic (say F_{01}) for H_{01}, and then rejecting when F_{01} is too large. Appropriate critical values are given in Dickey and Fuller (1981); see also Hamilton (1994, Table B.7, Case 2).

1.3. AR(1) model with trend

Model :

$$Y_t = \mu_0 + \mu_1 t + \varphi Y_{t-1} + u_t,\ t = 1, \ldots, T, \tag{1.13}$$

$$Y_0 \text{ is fixed,} \tag{1.14}$$

$$(u_t : t = 1, \ldots, T) \sim IID\left(0, \sigma^2\right). \tag{1.15}$$

$$H_0 : \varphi = 1, \tag{1.16}$$

$$H_{01} : \varphi = 1 \text{ and } \mu_1 = 0, \tag{1.17}$$
$$H_{02} : \varphi = 1 \text{ and } \mu_0 = \mu_1 = 0. \tag{1.18}$$

Let $\hat{\varphi}$ be the OLS estimator of φ obtained from equation (1.13) and let $t_{\varphi-1}$ the t statistic associated with H_0. Again we can test H_0 with $T\left(\hat{\varphi}-1\right)$ or $t_{\varphi-1}$. For the case where the u_t's are normal, critical points are given by Fuller (1976, pp. 371 and 373): Table 8.5.1 $(\hat{\rho}_\tau)$ and Table 8.5.2 $(\hat{\tau}_\tau)$ for $t_{\varphi-1}$. These critical points are also asymptotically valid (with $T=\infty$) when the u_t's are not normal.

For testing H_{01} and H_{02}, we can use the corresponding Fisher statistics. Tables are available from Dickey and Fuller (1981); for H_{02}, see Hamilton (1994, Table B.7, Case 4).

2. Unit root tests in AR(p) models

2.1. AR(p) model without drift

Model:

$$Y_t = \sum_{j=1}^{p} \varphi_j Y_{t-j} + u_t,\ t = p+1, \ldots, T, \tag{2.1}$$
$$(u_t : t = 1, \ldots, T) \sim IID\left(0, \sigma^2\right), \tag{2.2}$$

$$\begin{gathered}\varphi\left(B\right) \equiv 1 - \varphi_1 B - \varphi_2 B^2 - \cdots - \varphi_p B^p \\ \text{has all its roots outside the unit circle} \\ \text{except possibly one which can be equal to } 1.\end{gathered} \tag{2.3}$$

Hypothesis:

$$H_0 : \varphi\left(1\right) = 0 \tag{2.4}$$

or equivalently,

$$H_0 : \sum_{j=1}^{p} \varphi_j = 1. \tag{2.5}$$

Under H_0, we can write

$$\varphi\left(B\right) = \left(1 - B\right)\varphi_1\left(B\right) \tag{2.6}$$

where the polynomial $\varphi_1(B)$ has all its roots outside the unit circle. Equation (2.1) can be rewritten in the form

$$\begin{aligned} Y_t &= \rho_1 Y_{t-1} + \sum_{j=2}^{p} \rho_j \left(Y_{t-j+1} - Y_{t-j}\right) + u_t \\ &= \rho_1 Y_{t-1} + \sum_{j=1}^{p-1} \rho_{j+1} \left(Y_{t-j} - Y_{t-j-1}\right) + u_t \\ &= \rho_1 Y_{t-1} + \sum_{j=1}^{p-1} \rho_{j+1} \Delta Y_{t-j} + u_t\,,\ t = p+1, \ldots, T, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \rho_j &= \sum_{i=1}^{p} \varphi_i\,, \quad \text{for } j = 1\,, \\ &= -\sum_{i=j}^{p} \varphi_i\,, \quad \text{for } j = 2,\, \ldots,\, p\,, \end{aligned} \tag{2.8}$$

and $\Delta = 1 - B$. Under H_0, $\rho_1 = 1$. For testing H_0, we can use Student's t statistic $t_{\rho_1 - 1}$ associated with H_0 from the OLS estimation of equation (2.7). Another way of computing $t_{\rho_1 - 1}$ would consist in considering the equivalent equation

$$\Delta Y_t = \overline{\rho}_1 Y_{t-1} + \sum_{j=1}^{p-1} \rho_{j+1} \Delta Y_{t-j} + u_t\,,\ t = p+1, \ldots, T, \tag{2.9}$$

where $\overline{\rho}_1 = \rho_1 - 1$, from which it is easy to see that the t statistic $t_{\overline{\rho}_1}$ for testing

$$H_0' : \overline{\rho}_1 = 0 \tag{2.10}$$

in (2.9) is identical to $t_{\rho_1 - 1}$ obtained from (2.7).

From a distributional point of view, the main observation to be made here is that the asymptotic distribution of $t_{\overline{\rho}_1}$ is not affected by the presence of the additional lagged regressors in (2.9) with respect to . The relevant critical points are available in Fuller (1976, Table 8.5.2, $\hat{\tau},\ n = \infty$). These critical points are only valid asymptotically.

The equivalence between (2.1) and (2.7) can be shown as follows:

$$Y_t = \left(\sum_{j=1}^{p} \varphi_j\right) Y_{t-1} + \sum_{j=2}^{p} \left(-\sum_{i=j}^{p} \varphi_i\right) \left(Y_{t-j+1} - Y_{t-j}\right) + u_t$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{p}\varphi_j\right)Y_{t-1} - \sum_{j=2}^{p}\sum_{i\geq j}\varphi_i\left(Y_{t-j+1}-Y_{t-j}\right)+u_t \\
&= \left(\sum_{j=1}^{p}\varphi_j\right)Y_{t-1} - \sum_{i=2}^{p}\sum_{j=2}^{i}\varphi_i\left(Y_{t-j+1}-Y_{t-j}\right)+u_t \\
&= \left(\sum_{j=1}^{p}\varphi_j\right)Y_{t-1} - \sum_{i=2}^{p}\varphi_i\sum_{j=2}^{i}\left(Y_{t-j+1}-Y_{t-j}\right)+u_t \\
&= \left(\sum_{j=1}^{p}\varphi_j\right)Y_{t-1} - \sum_{i=2}^{p}\varphi_i\left(Y_{t-1}-Y_{t-i}\right)+u_t \\
&= \left(\sum_{j=1}^{p}\varphi_j\right)Y_{t-1} - \sum_{j=2}^{p}\varphi_j\left(Y_{t-1}-Y_{t-j}\right)+u_t \\
&= \left(\sum_{j=1}^{p}\varphi_j\right)Y_{t-1} - \left(\sum_{j=2}^{p}\varphi_j\right)Y_{t-1}+\sum_{j=2}^{p}\varphi_jY_{t-j}+u_t \\
&= \sum_{j=1}^{p}\varphi_jY_{t-j}+u_t. \qquad (2.11)
\end{aligned}
$$

2.2. AR(p) model with drift

Model:

$$Y_t = \mu_0 + \sum_{j=1}^{p}\varphi_jY_{t-j}+u_t\,,\ t=p+1,\ldots,T, \qquad (2.12)$$

with (2.2) and (2.3).
Hypothesis:

$$H_0 : \sum_{j=1}^{p}\varphi_j = 1. \qquad (2.13)$$

As for (2.1), model (2.12) can be rewritten in the form

$$\mathit{\Delta}Y_t = \mu_0 + \overline{\rho}_1Y_{t-1}+\sum_{j=1}^{p-1}\rho_{j+1}\mathit{\Delta}Y_{t-j}+u_t\,,\ t=p+1,\ldots,T, \qquad (2.14)$$

where $\overline{\rho}_1 = \rho_1 - 1$ and $\rho_1 = \sum_{j=1}^{p} \varphi_j$. H_0 may then be tested by testing

$$H_0' : \overline{\rho}_1 = 0 \tag{2.15}$$

in (2.14) using the corresponding OLS t statistic.

If it is assumed that $\mu_0 = 0$, the appropriate tables are those for $\hat{\tau}_\mu$ in Fuller (1976, Table 8.5.2). If $\mu_0 \neq 0$, the $N(0,1)$ distribution yields asymptotically valid critical values for $\hat{\tau}_\mu$. However, when it is not known whether $\mu_0 = 0$ or $\mu_0 \neq 0$, one should use the most conservative critical value: in the case of left-tailed one sided tests, this leads to employ the critical values applicable when $\mu_0 = 0$.

Similarly, the $F-$test of

$$H_{01} : \overline{\rho}_1 = 0 \text{ and } \mu_0 = 0 \tag{2.16}$$

can be performed done by computing the usual $F-$statistic (say F_{01}) for H_{01} based on (2.12), and then rejecting when F_{01} is too large. The appropriate critical values are the same as for the AR(1) case ; see Dickey and Fuller (1981) or Hamilton (1994, Table B.7, Case 2).

2.3. AR(p) model with trend

Model:

$$Y_t = \mu_0 + \mu_1 t + \sum_{j=1}^{p} \varphi_j Y_{t-j} + u_t,\ t = p+1, \dots, T, \tag{2.17}$$

with (2.2) and (2.3).
Hypothesis:

$$H_0 : \sum_{j=1}^{p} \varphi_j = 1. \tag{2.18}$$

As in the previous cases, equation (2.17) can be rewritten in the form

$$\Delta Y_t = \mu_0 + \mu_1 t + \overline{\rho}_1 Y_{t-1} + \sum_{j=1}^{p-1} \rho_{j+1} \Delta Y_{t-j} + u_t\ ,\ t = p+1, \dots, T. \tag{2.19}$$

H_0 may then be tested by testing $\overline{\rho}_1 = 0$ using the OLS t statistic obtained by OLS estimation of equation (2.19).

If it is assumed that $\mu_1 = 0$, the appropriate tables are those for $\hat{\tau}_\tau$ in Fuller (1976, Table 8.5.2). If $\mu_1 \neq 0$, the $N(0,1)$ distribution yields asymptotically valid critical values for $\hat{\tau}_\tau$. However, when it is not known whether $\mu_1 = 0$ or $\mu_1 \neq 0$, one should use the most conservative critical value: in the case of left-tailed one sided tests, this leads one to employ the critical values applicable when $\mu_1 = 0$.

Similarly, for the joint hypothesis

$$H_{01} : \overline{\rho}_1 = 1 \text{ and } \mu_1 = 0\,, \tag{2.20}$$

we can use the corresponding $F-$statistic based on (2.19) and the critical values given by Dickey and Fuller (1981); for H_{02}, see Hamilton (1994, Table B.7, Case 4).

3. Bibliographic notes

For further discussion of unit root tests, the reader may consult: Said and Dickey (1985), Schmidt (1990), Dufour and King (1991), Mills (1993, Chapter 3), Enders (1995, Chapter 4), Hamilton (1994, Chapters 15, 16, 17), Fuller (1996, Chapter 10), Tanaka (1996), Dufour and King (1991), Kiviet and Dufour (1997), Dufour and Kiviet (1998), Maddala and Kim (1998).

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