Estimation of the mean and autocorrelations of a stationary process [*]

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First version: December 1998
This version: May 3, 2015, 5:52pm

[*] This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), the Toulouse School of Economics (Pierre-de-Fermat Chair of excellence), the Universitad Carlos III de Madrid (Banco Santander de Madrid Chair of excellence), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).2

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Contents

List of Definitions, Assumptions, Propositions and Theorems

1. General distributional results

1.1 Suppose we have T observations $X_1, X_2, \ldots, X_T$ from a realization of a second-order stationary process. The natural estimators of the first and second moments of the process are: for the mean,

$$\overline{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t \,,$$

for the autocovariances

$$c_k = \frac{1}{T}\sum_{t=1}^{T-k}\left(X_t - \overline{X}_T\right)\left(X_{t+k} - \overline{X}_T\right)\,, \quad 1 \le k \le T-1\,,$$

and for the autocorrelations

$$r_k = c_k / c_0\,, \quad 1 \le k \le T-1\,.$$

1.2 Theorem DISTRIBUTION OF THE ARITHMETIC MEAN. *Let $\{X_t : t \in \mathbb{Z}\}$ a second-order stationary process with mean μ, and let $\overline{X}_T = \sum_{t=1}^{T} X_t / T$. Then*

(1) *$E\left(\overline{X}_T\right) = \mu$ and $\overline{X}_T$ is an unbiased estimator of μ;*

(2) *$Var\left(\overline{X}_T\right) = \frac{1}{T}\sum_{k=-(T-1)}^{T-1}\left(1 - \frac{|k|}{T}\right)\gamma_x(k)$;*

(3) *if $\gamma_x(k) \underset{k\to\infty}{\longrightarrow} 0$,*

$$Var\left(\overline{X}_T\right) \underset{T\to\infty}{\longrightarrow} 0 \text{ and } \overline{X}_T \overset{m.q.}{\underset{T\to\infty}{\longrightarrow}} \mu\ ;$$

(4) *if the series $\sum_{k=-\infty}^{\infty}\gamma_x(k)$ converges, then*

$$\lim_{T\to\infty} T\ Var\left(\overline{X}_T\right) = \sum_{k=-\infty}^{\infty}\gamma_x(k)\ ;$$

(5) *if the spectral density $f_x(\omega)$ exists and is continuous at $\omega = 0$, then*

$$\lim_{T\to\infty} T\ Var\left(\overline{X}_T\right) = 2\pi f_x(0)\ ;$$

(6) *if*

$$X_t = \mu + \sum_{j=-\infty}^{\infty}\psi_j u_{t-j}\,,\ \text{where } \{u_t : t \in \mathbb{Z}\} \sim IID\left(0, \sigma^2\right)\,,$$

and

$$\sum_{j=-\infty}^{\infty}|\psi_j| < \infty\,,$$

then

$$\sqrt{T}\left(\overline{X}_T - \mu\right) \overset{L}{\underset{T\to\infty}{\longrightarrow}} N\left[0, \sum_{k=-\infty}^{\infty}\gamma_x(k)\right]$$

and

$$\sum_{k=-\infty}^{\infty} \gamma_x(k) = \sigma^2 \left(\sum_{j=-\infty}^{\infty} \psi_j \right)^2 .$$

PROOF. See Anderson (1971, Sections 8.3.1 and 8.4.1) and Brockwell and Davis (1991, Section 7.1). □

1.3 Theorem DISTRIBUTION OF SAMPLE AUTOCORRELATIONS FOR A LINEAR STATIONARY PROCESS. *Let $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$, where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\{u_t : t \in \mathbb{Z}\} \sim IID\left(0, \sigma^2\right)$. If*

$$(a) \sum_{j=-\infty}^{\infty} |j| \psi_j^2 < \infty$$

or

$$(b)\; E\left(u_t^4\right) < \infty,\; \forall t\,,$$

then the asymptotic distribution of the vector

$$\left[\sqrt{T}\left(r_1 - \rho_1\right), \sqrt{T}\left(r_2 - \rho_2\right), \ldots, \sqrt{T}\left(r_m - \rho_m\right)\right]'$$

is $N[0, W_m]$ as $T \to \infty$, where $\rho_k = \gamma_x(k)/\gamma_x(0)$, $W_m = \left[w_{jk}\right]_{j,\, k=1,\ldots,\, m}$ and

$$\begin{aligned}
w_{jk} &= \sum_{h=-\infty}^{\infty} \left(\rho_{h+j}\rho_{h+k} + \rho_{h-j}\rho_{h+k} - 2\,\rho_k\rho_h\rho_{h+j} - 2\,\rho_j\rho_h\rho_{h+k} + 2\,\rho_j\rho_k\rho_h^2\right) \\
&= \sum_{h=1}^{\infty} \left(\rho_{h+j} + \rho_{h-j} - 2\,\rho_j\rho_h\right)\left(\rho_{h+k} + \rho_{h-k} - 2\,\rho_k\rho_h\right) \\
&= \frac{4\pi}{\gamma_x(0)^2} \int_{-\pi}^{\pi} \left[\cos(\omega j) - \rho_j\right]\left[\cos(\omega k) - \rho_k\right] f_x(\omega)^2\, d\omega\,.
\end{aligned}$$

PROOF. See Anderson (1971, Theorem 8.4.6, p. 489) and Brockwell and Davis (1991, Theorems 7.2.1 and 7.2.2). □

1.4 The expressions w_{jk} are called Bartlett's formula for the covariances of the autocorrelations. The formula w_{jk} may also be written

$$\begin{aligned}
w_{jk} &= \left(\lambda_{j+k} + \lambda_{j-k} - 2\rho j \lambda_k - 2\rho_k\lambda_j + 2\rho_j\rho_k\lambda_0\right)/\gamma_0^2 \\
&= \bar{\lambda}_{j+k} + \bar{\lambda}_{j-k} - 2\rho_j\bar{\lambda}_k - 2\rho_k\bar{\lambda}_j - 2\rho_j\rho_k\bar{\lambda}_0
\end{aligned}$$

where

$$\lambda_i = \sum_{h=-\infty}^{\infty} \gamma_h\gamma_{h+i}\,, \quad \bar{\lambda}_i \equiv \lambda_i/\gamma_0^2 = \sum_{h=-\infty}^{\infty} \rho_h\rho_{h+i}\,.$$

2. Special cases

2.1 ASYMPTOTIC VARIANCE. Under the conditions of Theorem **1.3**, the asymptotic distribution of $\sqrt{T}\,(r_k - \rho_k)$ is $N[0, w_{kk}]$, where

$$\begin{aligned} w_{kk} &= \sum_{h=-\infty}^{\infty} \left(\rho_{h+k}^2 + \rho_{h-k}\rho_{h+k} - 4\,\rho_k\rho_h\rho_{h+k} + 2\,\rho_k^2\rho_h^2\right) \\ &= \sum_{h=-\infty}^{\infty} \left(\rho_h^2 + \rho_h\rho_{h+2k} - 4\,\rho_h\rho_k\rho_{h+k} + 2\,\rho_h^2\rho_k^2\right) \\ &= \sum_{h=1}^{\infty} \left(\rho_{h+k} + \rho_{h-k} - 2\,\rho_h\rho_k\right)^2 . \end{aligned}$$

For T large, $\sqrt{T}\,(r_k - \rho_k) \overset{a}{\sim} N[0, w_{kk}]$.

2.2 WHITE NOISE. If

$$\begin{aligned} \rho_k &= 1\,,\ \text{for } k = 0\,, \\ &= 0\,,\ \text{for } k \neq 0\,, \end{aligned}$$

we find

$$\begin{aligned} w_{jk} &= 1\,,\ \text{if } j = k \\ &= 0\,,\ \text{if } j \neq k\,. \end{aligned}$$

For T large, the sampling autocorrelations are mutually uncorrelated and

$$\sqrt{T} r_k \overset{a}{\sim} N[0,1]\ ,\ \text{for } k \geq 1\,.$$

2.3 MA(q) PROCESS. If $\rho_k = 0$, for $|k| \geq q+1$, we find

$$\begin{aligned} w_{jk} &= \sum_{h=1}^{\infty} \rho_{h-j}\rho_{h-k} = \sum_{h=1}^{\infty} \rho_{j-h}\rho_{k-h} = \sum_{h=1}^{\infty} \rho_{k-h+(j-k)}\rho_{k-h} \\ &= \sum_{h=-\infty}^{k-1} \rho_h\rho_{h+(j-k)} = \sum_{h=-q}^{q-(j-k)} \rho_h\rho_{h+(j-k)}\,,\ \text{for } j \geq k \geq q+1\,, \end{aligned}$$

hence

$$\begin{aligned} w_{jk} &= 0\,, && \text{if } k \geq q+1 \text{ and } j \geq k+2q+1 \\ &= \textstyle\sum_{h=-q}^{q-(j-k)} \rho_h\rho_{h+(j-k)}\,, && \text{if } q+1 \leq k \leq j \leq k+2q\,. \end{aligned} \tag{2.1}$$

In particular,

$$w_{kk} = \sum_{h=-q}^{q} \rho_h^2 = 1 + 2\sum_{h=1}^{q} \rho_h^2\,,\ \text{if } k \geq q+1\,.$$

3. Exact tests of randomness

3.1 Theorem EXACT MOMENTS OF AUTOCORRELATIONS FOR AN *i.i.d.* SAMPLE. *Let the random variables $X_1, \ldots, X_T$ be independent and identically distributed $(i.i.d.)$ according to a continuous distribution. Then*

$$E(r_k) = -\frac{T-k}{T(T-1)} \text{ , for } 1 \leq k \leq T-1 \text{ ,}$$

and

$$Var(r_k) \leq \overline{V}_k \text{ ,}$$

where

$$\overline{V}_k \equiv \frac{T^4 - (k+7)T^3 + (7k+16)T^2 + 2\left(k^2 - 9k - 6\right)T - 4k(k-4)}{T(T-1)^2(T-2)(T-3)}$$

if $1 \leq k < T/2$ *and* $T > 3$, *and*

$$\overline{V}_k \equiv \frac{(T-k)\left[T^2 - 3T - 2(k-2)\right]}{T(T-1)^2(T-3)}$$

if $T/2 \leq k < T$ *and* $T > 3$.

PROOF. See Dufour and Roy (1985). □

3.2 For $k = 1$, we find

$$E(r_1) = -1/T \text{ ,}$$

$$Var(r_1) \leq \frac{T-2}{T(T-1)} \text{ .}$$

By Chebyshev's inequality,

$$P\left[|r_k - E(r_k)| \geq \lambda\right] \leq \frac{Var(r_k)}{\lambda^2} \leq \frac{\overline{V}_k}{\lambda^2} \text{ .}$$

3.3 Theorem EXACT MOMENTS OF AUTOCORRELATIONS FOR A GAUSSIAN *i.i.d.* SAMPLE. *Let $X_1, \ldots, X_T$ be i.i.d. random variables following a distribution $N\left[\mu, \sigma^2\right]$ distribution. Then*

$$E(r_k) = -\frac{(T-k)}{T(T-1)} \text{ , for } 1 \leq k \leq T-1 \text{ ,}$$

$$Var(r_k) = \frac{T^4 - (k+3)T^3 + 3kT^2 + 2k(k+1)T - 4k^2}{(T+1)T^2(T-1)^2}$$

for $1 \leq k < T/2$ *and* $T > 3$ *,and*

$$Var(r_k) = \frac{(T-k)(T-2)\left(T^2 + T - 2k\right)}{(T+1)T^2(T-1)^2}$$

for $T/2 \le k < T$ *and* $T > 3$ *. Furthermore, for* $1 \le k < h \le T-1$ *,*

$$Cov(r_k, r_h) = \frac{2\left[kh(T-1) - (T-h)\left(T^2-k\right)\right]}{(T+1)\,T^2\,(T-1)^2}$$

if $l < h+k < T$ *, and*

$$Cov(r_k, r_h) = \frac{2(T-h)\left[2k-(k+1)T\right]}{(T+1)\,T^2\,(T-1)^2}$$

if $h+k \ge T$ *.*

PROOF. See Dufour and Roy (1985). □

3.4 For T large, we have

$$\frac{r_k - E(r_k)}{\left[Var(r_k)\right]^{1/2}} \overset{a}{\sim} N(0,1) \ .$$

In small or moderately large samples, the normal approximation is much more accurate when the formulae for $E(r_k)$ et $Var(r_k)$ given by Theorem **3.3** are used, rather than $E(r_k) = 0$ et $Var(r_k) = 1/T$; see Dufour and Roy (1985).

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