

Linear models with nonscalar covariance matrix and generalized least squares *

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1. Generalized least squares

1.1. Best linear unbiased estimator

$$y = X\beta + u \quad (1.1)$$

where y is a $T \times 1$ vector of observations on a dependent variable, X is a $T \times k$ nonstochastic matrix of rank k , and u is a $T \times 1$ vector of disturbances (errors) such that

$$\begin{aligned} E(u) &= 0 \\ V(u) &= \sigma^2 V \end{aligned} \quad (1.2)$$

and V is a known $T \times T$ positive definite matrix. Then the least-squares estimator

$$\hat{\beta} = (X'X)^{-1}X'y \quad (1.3)$$

is unbiased but does not have minimal variance. The covariance matrix of $\hat{\beta}$ is

$$V(\hat{\beta}) = \sigma^2(X'X)^{-1}X'VX(X'X)^{-1} \quad (1.4)$$

so that the usual formula

$$V(\hat{\beta}) = \sigma^2(X'X)^{-1} \quad (1.5)$$

is not valid.

The fact V is positive definite entails that $|V| \neq 0$, so there is no perfect correlation between the disturbances. Further, there exists a nonsingular $T \times T$ matrix P such that

$$PVP' = I_T, \quad (1.6)$$

$$(P')^{-1}V^{-1}P^{-1} = (PVP')^{-1} = I_T. \quad (1.7)$$

Multiply both sides of (1.1) by P :

$$Py = PX\beta + Pu. \quad (1.8)$$

We get in this way the transformed model

$$y_* = X_*\beta + u_* \quad (1.9)$$

where

$$y_* = Py, \quad X_* = PX, \quad u_* = Pu \quad (1.10)$$

$$E(u_*) = 0, \quad (1.11)$$

$$V(u_*) = E[Pu u' P'] = \sigma^2 P V P' = \sigma^2 I_T. \quad (1.12)$$

Then

$$\hat{\beta}_G = (X_*' X_*)^{-1} X_*' y_* \quad (1.13)$$

is the best linear unbiased estimator of β :

$$\begin{aligned} E(\hat{\beta}_G) &= \beta \\ V(\hat{\beta}_G) &= \sigma^2 (X_*' X_*)^{-1}. \end{aligned} \quad (1.14)$$

We can also write:

$$\hat{\beta}_G = (X' P' P X)^{-1} X' P' P y = (X' V^{-1} X)^{-1} X' V^{-1} y \quad (1.15)$$

for

$$\begin{aligned} P V P' = I_T &\Rightarrow V = P^{-1} (P')^{-1} = (P' P)^{-1} \\ &\Rightarrow V^{-1} = P' P. \end{aligned} \quad (1.16)$$

$\hat{\beta}_G$ is called the generalized least squares estimator of β :

$$\begin{aligned} E(\hat{\beta}_G) &= \beta, \\ V(\hat{\beta}_G) &= \sigma^2 (X_*' X_*)^{-1} = \sigma^2 (X' V^{-1} X)^{-1}. \end{aligned} \quad (1.17)$$

We know that $\hat{\beta}$ minimizes

$$(y - X\beta)' (y - X\beta). \quad (1.18)$$

Similarly, $\hat{\beta}_G$ minimizes

$$\begin{aligned} (y_* - X_*\beta)' (y_* - X_*\beta) &= (P y - P X \beta)' (P y - P X \beta) \\ &= (y - X \beta)' P' P (y - X \beta) \\ &= (y - X \beta)' V^{-1} (y - X \beta) \end{aligned}$$

This is why $\hat{\beta}_G$ is also called a *weighted least squares* estimator of β .

1.2. Gaussian case

Suppose

$$u \sim N [0, \sigma^2 V] \quad (1.19)$$

Then

$$\hat{\beta}_G \sim N \left[\beta, \sigma^2 (X' V^{-1} X)^{-1} \right] \quad (1.20)$$

is the best mean squares unbiased estimator of β .

We can build tests and confidence intervals in the usual manner by using the transformed model

$$(Py) = (PX)\beta + (Pu) \quad (1.21)$$

instead of

$$y = X\beta + u. \quad (1.22)$$

2. Estimation with heteroskedasticity

2.1. Known variance structure

Suppose

$$E[uu'] = \sigma^2 \begin{bmatrix} d_1^2 & 0 & \cdots & 0 \\ 0 & d_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_T^2 \end{bmatrix} = \sigma^2 V. \quad (2.1)$$

The variance of each element of u is then

$$V(u_t) = \sigma_t^2 = d_t^2 \sigma^2 \quad (2.2)$$

and we have:

$$y_t = x_t' \beta + u_t, \quad t = 1, \dots, T$$

$$\frac{y_t}{d_t} = \frac{1}{d_t} x_t' \beta + \frac{u_t}{d_t}, \quad t = 1, \dots, T \quad (2.3)$$

$$y_{*t} = x_{*t}' \beta + u_{*t}, \quad t = 1, \dots, T \quad (2.4)$$

$$V(u_{*t}^2) = V\left(\frac{u_t}{d_t}\right) = \sigma^2 \frac{d_t^2}{d_t^2} = \sigma^2 \quad (2.5)$$

$$P = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_T \end{bmatrix} \quad (2.6)$$

2.2. Unknown variance structure

It is rare that d_1, \dots, d_T are known.

It is impossible to estimate $T + k$ parameters with T observations (incidental parameter problem)..

One must make hypotheses on the form of the variance structure.

1. $d_t^2 = c(x_{tk})^2$
where x_k is one of the explanatory variables or another variable. Then

$$\frac{y_t}{x_{tk}} = \frac{1}{x_{tk}} x_t' \beta + \frac{u_t}{x_{tk}}, \quad t = 1, \dots, T$$

$$V\left(\frac{u_t}{x_{tk}}\right) = \sigma^2 c = c\sigma^2 \quad (2.7)$$

$$2. \sigma_t^2 = c (\mathbb{E}y_t)^2 = c (x_t'\beta)^2$$

Then

$$\frac{y_t}{\mathbb{E}(y_t)} = \frac{1}{\mathbb{E}(y_t)} x_t'\beta + \frac{u_t}{\mathbb{E}(y_t)}, \quad t = 1, \dots, T \quad (2.8)$$

A difficulty here is that $\mathbb{E}(y_t) = x_t'\beta$ is unknown. This suggests a two-step procedure.

1. Estimate β par OLS. This is reasonable because $\hat{\beta}$ is unbiased.
2. The model is then transformed according to:

$$\frac{y_t}{x_t'\hat{\beta}} = \left(\frac{1}{x_t'\hat{\beta}} x_t' \right) \beta + \frac{u_t}{x_t'\hat{\beta}}. \quad (2.9)$$

In this way, the model becomes “approximately homoskedastic”. For T large this leads to efficient estimators and valid tests and confidence intervals.