

Basic asymptotic theory ^{*}

Jean-Marie Dufour [†]
McGill University

First version: November 2011

Revised: December 2011

This version: December 2011

Compiled: December 12, 2011, 22:25

^{*}This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Canadian Network of Centres of Excellence [program on *Mathematics of Information Technology and Complex Systems* (MITACS)], the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

[†]William Dow Professor of Economics, McGill University, Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Centre interuniversitaire de recherche en économie quantitative (CIREQ). Mailing address: Department of Economics, McGill University, Leacock Building, Room 519, 855 Sherbrooke Street West, Montréal, Québec H3A 2T7, Canada. TEL: (1) 514 398 8879; FAX: (1) 514 398 4938; e-mail: jean-marie.dufour@mcgill.ca . Web page: <http://www.jeanmariedulfour.com>

Contents

1. Stochastic convergence	1
1.1. Basic definitions	1
1.2. Relations between convergence concepts	2
1.3. Convergence of expectations and functions of random variables .	3
1.4. Random series	4
2. Laws of large numbers	5
3. Central limit theorems	6
3.1. Extension to random vectors	7
3.2. Proofs and additional references	9

1. Stochastic convergence

1.1. Basic definitions

1.1 Definition Let $\{X_n = X_n(\omega) : n = 1, 2, \dots\}$ a sequence of real r.v.'s defined on a probability space (Ω, \mathcal{A}, P) and $X = X(\omega)$ another real r.v. defined on the same space.

(a) X_n converges in probability to X as $n \rightarrow \infty$ (denoted $X_n \xrightarrow{P} X$) iff

$$\lim_{n \rightarrow \infty} P[|X_n - X| > \varepsilon] = 0, \forall \varepsilon > 0. \quad (1.1)$$

(b) X_n converges almost surely to X as $n \rightarrow \infty$ (denoted $X_n \xrightarrow{a.s.} X$) iff

$$P\left[\lim_{n \rightarrow \infty} X_n = X\right] = 1. \quad (1.2)$$

(c) Suppose $E|X_n|^r < \infty, \forall n$, where $r > 0$. X_n converges in mean of order r to X (denoted $X_n \xrightarrow{r} X$) iff

$$\lim_{n \rightarrow \infty} E[|X_n - X|^r] = 0. \quad (1.3)$$

In this case, we also say that X_n converges to X in L_r . If $r = 2$, we say X_n converges to X in quadratic mean (q.m.).

(d) Let $F_n(x)$ and $F(x)$ be the distribution functions of X_n and X respectively. X_n converges in law (or in distribution) to X as $n \rightarrow \infty$ (denoted $X_n \xrightarrow{L} X$) iff

$$\lim_{n \rightarrow \infty} F_n(x) = F(x) \text{ at all continuity points of } F(x). \quad (1.4)$$

An important special case of the above concepts is the one where X is a fixed real constant c .

1.2 Definition Let $\{X_n = X_n(\omega) : n = 1, 2, \dots\}$ a sequence of real r.v.'s defined on a probability space (Ω, \mathcal{A}, P) and c a real constant.

(a) X_n converges in probability to X as $n \rightarrow \infty$ (denoted $X_n \xrightarrow{p} c$) iff

$$\lim_{n \rightarrow \infty} P[|X_n - c| > \varepsilon] = 0, \forall \varepsilon > 0. \quad (1.5)$$

(b) X_n converges almost surely to c as $n \rightarrow \infty$ (denoted $X_n \xrightarrow{a.s.} c$) iff

$$P\left[\lim_{n \rightarrow \infty} X_n = c\right] = 1. \quad (1.6)$$

(c) Suppose $E|X_n|^r < \infty, \forall n$, where $r > 0$. X_n converges in mean of order r to c (denoted $X_n \xrightarrow{r} c$) iff

$$\lim_{n \rightarrow \infty} E[|X_n - c|^r] = 0. \quad (1.7)$$

In this case, we also say that X_n converges to c in L_r . If $r = 2$, we say X_n converges to c in quadratic mean (*q.m.*).

1.3 Proposition UNICITY OF PROBABILITY LIMIT. Let $\{X_n : n = 1, 2, \dots\}$ be a sequence of real r.v.'s defined on a probability space (Ω, \mathcal{A}, P) , and let X and Y be two real r.v.'s defined on the same probability space. Then

$$X_n \xrightarrow{p} X \text{ and } X_n \xrightarrow{p} Y \Rightarrow P[X \neq Y] = 0. \quad (1.8)$$

1.2. Relations between convergence concepts

1.4 Assumption Let $\{X_n\} \equiv \{X_n : n = 1, 2, \dots\}$ be a sequence of real r.v.'s defined on a probability space (Ω, \mathcal{A}, P) and X another real r.v. defined on the same space.

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.5 Proposition RELATIONS BETWEEN CONVERGENCE CONCEPTS.

$$(a) X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X.$$

$$(b) X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{s} X \text{ for all } s \text{ such that } 0 < s \leq r \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X.$$

1.6 Remark In general, the implications in **1.5** (b) and (c) cannot be reversed.

1.3. Convergence of expectations and functions of random variables

1.1 Assumption Let $\{X_n : n = 1, 2, \dots\}$ be a sequence of real r.v.'s, X a real r.v. and $g : \mathbb{R} \rightarrow \mathbb{R}$ a function such that $g(X)$ and $g(X_n)$, $n = 1, 2, \dots$, are real r.v.'s.

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.2 Proposition Let $g : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function everywhere on \mathbb{R} , except possibly in a set $A \subseteq \mathbb{R}$, and let X a r.v. such that $P[X \in A] = 0$. Then

$$(a) X_n \xrightarrow{p} X \Rightarrow g(X_n) \xrightarrow{p} g(X);$$

$$(b) X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X);$$

$$(c) X_n \xrightarrow{L} X \Rightarrow g(X_n) \xrightarrow{L} g(X).$$

1.3 Proposition Let $\{X_n\}$ and $\{Y_n\}$ two sequences of random variables. Then

$$(a) X_n \xrightarrow{p} X \text{ and } Y_n \xrightarrow{p} Y \Rightarrow X_n + Y_n \xrightarrow{p} X + Y;$$

$$(b) X_n \xrightarrow{a.s.} X \text{ and } Y_n \xrightarrow{a.s.} Y \Rightarrow X_n + Y_n \xrightarrow{a.s.} X + Y;$$

$$(c) X_n \xrightarrow{p} X \text{ and } Y_n \xrightarrow{p} Y \Rightarrow g(X_n, Y_n) \xrightarrow{p} g(X, Y) \text{ for any continuous function } g(x, y).$$

1.4 Proposition Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.'s such that $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{P} c$, where X is a r.v. and c is a real constant ($-\infty < c < +\infty$). Then

- (a) $X_n + Y_n \xrightarrow{L} X + c$;
- (b) $X_n Y_n \xrightarrow{L} Xc$;
- (c) $X_n/Y_n \xrightarrow{L} X/c$ if $c \neq 0$;
- (d) $(X_n, Y_n) \xrightarrow{L} (X, c)$.

1.5 Proposition Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.'s such that $X_n - Y_n \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y$, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then

- (a) $X_n \xrightarrow{L} Y$;
- (b) $g(X_n) - g(Y_n) \xrightarrow{P} 0$;
- (c) $g(X_n) \xrightarrow{L} g(Y)$.

1.4. Random series

1.6 Definition Let $\{X_t : t \in \mathbb{N}\}$ be a real-valued stochastic process and consider the series $\sum_{t=1}^{\infty} X_t$.

1.7 Definition We say $\sum_{t=1}^{\infty} X_t$ converges (according to given mode of convergence) iff there exists a real r.v. Y such that

$$\sum_{t=1}^N X_t \xrightarrow[N \rightarrow \infty]{} Y \text{ (according to the same mode of convergence) .}$$

1.8 Remark The mode of convergence : a.s., in probability or in mean of order r

2. Laws of large numbers

2.1 Proposition Let $\{X_t\}_{t=1}^{\infty}$ a sequence of r.v.'s such that $X_t \in L_2$ and $\text{Cov}(X_s, X_t) = 0$ for $s \neq t$, and let $\mu_t = E(X_t)$. Then

$$(a) \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Var}(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n \rightarrow \infty]{2} 0 \quad (\text{Chebychev law})$$

where $\bar{X}_n = \sum_{t=1}^n X_t/n$ and $\bar{\mu}_n = \sum_{t=1}^n \mu_t/n$, and

$$(b) \sum_{n=1}^{\infty} \left(\frac{\log n}{n} \right)^2 \text{Var}(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

In particular, if $\text{Var}(X_t) = \sigma^2 < \infty$ and $E(X_t) = \mu$ for all t , then

$$\frac{1}{n} \sum_{t=1}^n X_t \xrightarrow[n \rightarrow \infty]{a.s.} \mu \quad \text{and} \quad \frac{1}{n} \sum_{t=1}^n X_t \xrightarrow[n \rightarrow \infty]{2} \mu.$$

2.2 Theorem KHINTCHINE WEAK LAW OF LARGE NUMBERS. Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s whose mean $E(X_t)$ exists. Then

$$E(X_t) = \mu \Rightarrow \bar{X}_n \xrightarrow[n \rightarrow \infty]{p} \mu.$$

2.3 Theorem FIRST KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s such that $E(X_t) = \mu_t$ and $\text{Var}(X_t) = \sigma_t^2$ exist for all t . Then

$$\sum_{n=1}^{\infty} (\sigma_n/n)^2 < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

2.4 Theorem SECOND KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS.

Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s. Then

$$E(X_t) \text{ exists and is equal to } \mu \Leftrightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

3. Central limit theorems

3.1 Theorem LINDBERG-LÉVY CENTRAL LIMIT THEOREM. Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s in L_2 such that $E(X_t) = \mu$ and $Var(X_t) = \sigma^2 > 0$. Then

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n (X_t - \mu) / \sigma = \sqrt{n}(\bar{X}_t - \mu) / \sigma \xrightarrow[n \rightarrow \infty]{L} Z$$

where $Z \sim N(0, 1)$.

3.2 Theorem LIAPUNOV CENTRAL LIMIT THEOREM. Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s in L_3 such that $E(X_t) = \mu_t$, $Var(X_t) = \sigma_t^2 \neq 0$, $E[|X_t - \mu_t|^3] = \beta_t$ for all t . Moreover,

$$B_n = \left(\sum_{t=1}^n \beta_t \right)^{1/3}, C_n = \left(\sum_{t=1}^n \sigma_t^2 \right)^{1/2}.$$

If $\lim_{n \rightarrow \infty} (B_n / C_n) = 0$, then

$$\sum_{t=1}^n (X_t - \mu_t) / C_n = \sqrt{n}(\bar{X}_t - \bar{\mu}_n) / (C_n / \sqrt{n}) \sigma \xrightarrow[n \rightarrow \infty]{L} Z$$

where $Z \sim N(0, 1)$.

3.3 Theorem LINDBERG-FELLER CENTRAL LIMIT THEOREM. Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s in L_2 such that

$$P[X_t \leq x] = G_t(x), \quad E(X_t) = \mu_t, \quad \text{Var}(X_t) = \sigma_t^2 \neq 0,$$

for all t . Then

$$\sum_{t=1}^n (X_t - \mu_t) / C_n \xrightarrow[n \rightarrow \infty]{L} Z \text{ and } \lim_{n \rightarrow \infty} \max_{1 \leq t \leq n} (\sigma_t / C_n) = 0$$

iff

$$\lim_{n \rightarrow \infty} \frac{1}{C_n^2} \sum_{t=1}^n \int_{|x - \mu_t| > \varepsilon C_n} (x - \mu_t)^2 dG_t(x) = 0, \quad \forall \varepsilon > 0.$$

3.1. Extension to random vectors

3.1 Definition STOCHASTIC CONVERGENCE FOR VECTORS. Let $\{X_n\}_{n=1}^{\infty}$ a sequence of vectors of dimension k ,

$$X_n = (X_{1n}, X_{2n}, \dots, X_{kn})', \quad n = 1, 2, \dots$$

whose components are real random variables all defined on the same probability space (Ω, Q, P) , and

$$X = (X_1, X_2, \dots, X_k)'$$

another random vector of dimension k whose components are defined on the same space.

- (a) We say X_n converges to X in probability (almost surely, in mean of order r) as $n \rightarrow \infty$ if each component of X_n converges to the corresponding component of X in probability (almost surely, in mean of order r) as $n \rightarrow \infty$. Depending on the case considered, we then write $X_n \xrightarrow{P} X$, $X_n \xrightarrow{a.s.} X$ or $X_n \xrightarrow{r} X$.
- (b) We say X_n converges in law to X ($X_n \xrightarrow{L} X$) iff

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x) \text{ at all continuity points of } F_X(x).$$

where $x = (x_1, x_2, \dots, x_k)' \in \mathbb{R}^k$,

$$F_{X_n}(x) = P[X_{1n} \leq x_1, \dots, X_{kn} \leq x_k], \quad n = 1, 2, \dots$$

and

$$F_X(x) = P[X_1 \leq x_1, \dots, X_k \leq x_k].$$

3.2 Theorem UNIVARIATE CHARACTERIZATION OF CONVERGENCE IN LAW FOR A SEQUENCE OF VECTORS.. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random vectors of dimension $k \times 1$ and let X be another random vector of dimension $k \times 1$. Then

$$X_n \xrightarrow[n \rightarrow \infty]{L} X \Leftrightarrow \lambda' X_n \xrightarrow[n \rightarrow \infty]{L} \lambda' X, \quad \forall \lambda \in \mathbb{R}^k.$$

In particular, if $X \sim N[\mu, \Sigma]$,

$$X_n \xrightarrow[n \rightarrow \infty]{L} N[\mu, \Sigma] \Leftrightarrow \lambda' X_n \xrightarrow[n \rightarrow \infty]{L} N[\lambda' \mu, \lambda' \Sigma \lambda], \quad \forall \lambda \in \mathbb{R}^k.$$

3.2. Proofs and additional references

Proofs and further discussions of the results presented above may be found the following references: Rao (1973), Lukacs (1975), Stout (1974), Loève (1977).

References

- LOÈVE, M. (1977): *Probability Theory, Volumes I and II*. Springer-Verlag, New York, 4th edn.
- LUKACS, E. (1975): *Stochastic Convergence*. Academic Press, New York, second edn.
- RAO, C. R. (1973): *Linear Statistical Inference and its Applications*. John Wiley & Sons, New York, second edn.
- STOUT, W. F. (1974): *Almost Sure Convergence*. Academic Press, New York.