Basic asymptotic theory *

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1. Stochastic convergence

1.1. Basic definitions

1.1 Definition *Let* $\{X_n = X_n(\omega) : n = 1,\ 2,\ ...\}$ *a sequence of real r.v.'s defined on a probability space* $(\Omega,\ \mathscr{A},\ P)$ *and* $X = X(\omega)$ *another real r.v. defined on the same space.*

(*a*) X_n *converges* in probability *to* X *as* $n \to \infty$ $(\textit{denoted } X_n \xrightarrow{p} X)$ *iff*

$$\lim_{n\to\infty} P\left[|X_n - X| > \varepsilon\right] = 0\,,\ \forall \varepsilon > 0\,. \tag{1.1}$$

(*b*) X_n *converges* almost surely *to* X *as* $n \to \infty$ $(\textit{denoted } X_n \xrightarrow{a.s.} X)$ *iff*

$$P\left[\lim_{n\to\infty} X_n = X\right] = 1\,. \tag{1.2}$$

(*c*) *Suppose* $E|X_n|^r < \infty\,, \forall n$ *, where* $r > 0$*.* X_n *converges* in mean of order r *to* X $(\textit{denoted } X_n \xrightarrow{r} X)$ *iff*

$$\lim_{n\to\infty} E\left[|X_n - X|^r\right] = 0\,. \tag{1.3}$$

In this case, we also say that X_n *converges to* X *in* L_r*. If* $r = 2$ *, we say* X_n *converges to* X *in quadratic mean* $(q.m.)$*.*

(*d*) *Let* $F_n(x)$ *and* $F(x)$ *be the distribution functions of* X_n *and* X *respectively.* X_n *converges* in law (*or* in distribution) *to* X *as* $n \to \infty$ $(\textit{denoted } X_n \xrightarrow{L} X)$ *iff*

$$\lim_{n\to\infty} F_n(x) = F(x) \textit{ at all continuity points of } F(x)\,. \tag{1.4}$$

An important specila case of the above concepts is the one where X is a fixed real constant c.

1.2 Definition *Let* $\{X_n = X_n(\omega) : n = 1,\ 2,\ ...\}$ *a sequence of real r.v.'s defined on a probability space* $(\Omega,\ \mathscr{A},\ P)$ *and* c *a real constant.*

(*a*) X_n *converges* in probability *to* X *as* $n \to \infty$ $(\textit{denoted } X_n \xrightarrow{p} c)$ *iff*

$$\lim_{n\to\infty} P\left[|X_n - c| > \varepsilon\right] = 0\,,\ \forall \varepsilon > 0\,. \tag{1.5}$$

(*b*) X_n *converges* almost surely *to* c *as* $n \to \infty$ $(\textit{denoted } X_n \xrightarrow{a.s.} c)$ *iff*

$$P\left[\lim_{n\to\infty} X_n = c\right] = 1\,. \tag{1.6}$$

(*c*) *Suppose* $E|X_n|^r < \infty\,, \forall n$ *, where* $r > 0$*.* X_n *converges* in mean of order r *to* c $(\textit{denoted } X_n \xrightarrow{r} c)$ *iff*

$$\lim_{n\to\infty} E\left[|X_n - c|^r\right] = 0\,. \tag{1.7}$$

In this case, we also say that X_n converges to c in L_r. If $r = 2$, we say X_n converges to c in quadratic mean $(q.m.)$.

1.3 Proposition UNICITY OF PROBABILITY LIMIT. *Let $\{X_n : n = 1, 2, ...\}$ be a sequence of real r.v.'s defined on a probability space $(\Omega, \mathscr{A}, P)$, and let X and Y be two real r.v.'s defined on the same probability space. Then*

$$X_n \xrightarrow{p} X \text{ and } X_n \xrightarrow{p} Y \Rightarrow P[X \neq Y] = 0\,. \tag{1.8}$$

1.2. Relations between convergence concepts

1.4 Assumption *Let $\{X_n\} \equiv \{X_n : n = 1, 2, ...\}$ be a sequence of real r.v.'s defined on a probability space $(\Omega, \mathscr{A}, P)$ and X another real r.v. defined on the same space.*

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.5 Proposition RELATIONS BETWEEN CONVERGENCE CONCEPTS.

(a) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X$.

(b) $X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{s} X$ *for all s such that* $0 < s \leq r \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X$.

1.6 Remark In general, the implications in **1.5** (b) and (c) cannot be reversed.

1.3. Convergence of expectations and functions of random variables

1.1 Assumption *Let $\{X_n : n = 1, 2, ...\}$ be a sequence of real r.v.'s, X a real r.v. and $g : \mathbb{R} \to \mathbb{R}$ a function such that $g(X)$ and $g(X_n)$, $n = 1, 2, ...$, are real r.v.'s.*

Unless stated otherwise, this assumption will hold for all the definitions, propositions and theorems in this section.

1.2 Proposition *Let $g : \mathbb{R} \to \mathbb{R}$ a continuous function everywhere on $\mathbb{R}$, except possibly in a set $A \subseteq \mathbb{R}$, and let X a r.v. such that $P[X \in A] = 0$. Then*

(a) $X_n \xrightarrow{p} X \Rightarrow g(X_n) \xrightarrow{p} g(X)$;

(b) $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$;

(c) $X_n \xrightarrow{L} X \Rightarrow g(X_n) \xrightarrow{L} g(X)$.

1.3 Proposition *Let $\{X_n\}$ and $\{Y_n\}$ two sequences of random variables. Then*

(a) $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y \Rightarrow X_n + Y_n \xrightarrow{p} X + Y$;

(b) $X_n \xrightarrow{a.s.} X$ *and* $Y_n \xrightarrow{a.s.} Y \Rightarrow X_n + Y_n \xrightarrow{a.s.} X + Y$;

(c) $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} Y \Rightarrow g(X_n, Y_n) \xrightarrow{p} g(X, Y)$ *for any continuous function* $g(x,\ y)$.

1.4 Proposition *Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.'s such that $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{p} c$, where X is a r.v. and c is a real constant $(-\infty < c < +\infty)$. Then*

(a) $X_n + Y_n \xrightarrow{L} X + c$;

(b) $X_n Y_n \xrightarrow{L} Xc$;

(c) $X_n / Y_n \xrightarrow{L} X/c$ *if* $c \neq 0$;

(d) $(X_n, Y_n) \xrightarrow{L} (X, c)$.

1.5 Proposition *Let $\{X_n\}$ and $\{Y_n\}$ two sequences of r.v.'s such that $X_n - Y_n \xrightarrow{p} 0$ and $Y_n \xrightarrow{L} Y$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then*

(a) $X_n \xrightarrow{L} Y$;

(b) $g(X_n) - g(Y_n) \xrightarrow{p} 0$;

(c) $g(X_n) \xrightarrow{L} g(Y)$.

1.4. Random series

1.6 Definition *Let $\{X_t : t \in \mathbb{N}\}$ be a real-valued stochastic process and consider the series $\sum_{t=1}^{\infty} X_t$.*

1.7 Definition *We say $\sum_{t=1}^{\infty} X_t$ converges (according to given mode of convergence) iff there exists a real r.v. Y such that*

$$\sum_{t=1}^{N} X_t \underset{N\to\infty}{\longrightarrow} Y \text{ (according to the same mode of convergence)} .$$

1.8 Remark The mode of convergence : a.s., in probability or in mean of order r .

2. Laws of large numbers

2.1 Proposition *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of r.v.'s such that $X_t \in L_2$ and $Cov(X_s,\ X_t) = 0$ for $s \neq t$, and let $\mu_t = E(X_t)$. Then*

(a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} Var(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{2} 0$ *(Chebychev law)*

where $\bar{X}_n = \sum\limits_{t=1}^{n} X_t/n$ and $\bar{\mu}_n = \sum\limits_{t=1}^{n} \mu_t/n$, and

(b) $\sum\limits_{n=1}^{\infty} \left(\frac{\log n}{n}\right)^2 Var(X_n) < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0\,.$

In particular, if $Var(X_t) = \sigma^2 < \infty$ and $E(X_t) = \mu$ for all t, then

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow[n\to\infty]{a.s.} \mu \text{ and } \frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow[n\to\infty]{2} \mu\,.$$

2.2 Theorem KHINTCHINE WEAK LAW OF LARGE NUMBERS. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s whose mean $E(X_t)$ exists. Then*

$$E(X_t) = \mu \Rightarrow \bar{X}_n \xrightarrow[n\to\infty]{p} \mu\,.$$

2.3 Theorem FIRST KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s such that $E(X_t) = \mu_t$ and $Var(X_t) = \sigma_t^2$ exist for all t. Then*

$$\sum_{n=1}^{\infty} (\sigma_n/\ n)^2 < \infty \Rightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0\,.$$

2.4 Theorem SECOND KOLMOGOROV'S STRONG LAW OF LARGE NUMBERS. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s. Then*

$$E(X_t) \text{ exists and is equal to } \mu \Leftrightarrow \bar{X}_n - \bar{\mu}_n \xrightarrow[n\to\infty]{a.s.} 0\,.$$

3. Central limit theorems

3.1 Theorem LINDEBERG-LÉVY CENTRAL LIMIT THEOREM. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent and identically distributed r.v.'s in L_2 such that $E(X_t)=\mu$ and $Var(X_t)=\sigma^2>0$. Then*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(X_t-\mu)/\,\sigma=\sqrt{n}(\bar{X}_t-\mu)/\,\sigma\xrightarrow[n\to\infty]{L}Z$$

where $Z\sim N(0,1)$.

3.2 Theorem LIAPUNOV CENTRAL LIMIT THEOREM. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s in L_3 such that $E(X_t)=\mu_t$, $Var(X_t)=\sigma_t^2\neq 0$, $E[|X_t-\mu_t|^3]=\beta_t$ for all t. Moreover,*

$$B_n=\left(\sum_{t=1}^{n}\beta_t\right)^{1/3},C_n=\left(\sum_{t=1}^{n}\sigma_t^2\right)^{1/2}.$$

If $\lim_{n\to\infty}(B_n/\,C_n)=0$, then

$$\Sigma_{t=1}^{n}(X_t-\mu_t)/\,C_n=\sqrt{n}(\bar{X}_t-\bar{\mu}_n)/(C_n/\sqrt{n})\sigma\xrightarrow[n\to\infty]{L}Z$$

where $Z\sim N(0,1)$.

3.3 Theorem LINDEBERG-FELLER CENTRAL LIMIT THEOREM. *Let $\{X_t\}_{t=1}^{\infty}$ a sequence of independent r.v.'s in L_2 such that*

$$P[X_t\leq x]=G_t(x)\ ,\ E(X_t)=\mu_t\ ,\ Var(X_t)=\sigma_t^2\neq 0,$$

for all t. Then

$$\sum_{t=1}^{n}(X_t-\mu_t)/\,C_n\xrightarrow[n\to\infty]{L}Z \text{ and } \lim_{n\to\infty}\max_{1\leq t\leq n}(\sigma_t/\,C_n)=0$$

iff

$$\lim_{n\to\infty}\frac{1}{C_n^2}\sum_{t=1}^{n}\int_{|x-\mu_t|>\varepsilon C_n}(x-\mu_t)^2dG_t(x)=0,\forall\varepsilon>0.$$

3.1. Extension to random vectors

3.1 Definition STOCHASTIC CONVERGENCE FOR VECTORS. *Let $\{X_n\}_{n=1}^{\infty}$ a sequence of vectors of dimension k,*

$$X_n=(X_{1n},X_{2n},...,X_{kn})',n=1,2,...$$

whose components are real random variables all defined on the same probability space (Ω, Q, P), *and*

$$X = (X_1, X_2, ..., X_k)'$$

another random vector of dimension k *whose components are defined on the same space.*

(a) *We say* X_n *converges to* X *in probability* (*almost surely, in mean of order* r) *as* $n \to \infty$ *if each component of* X_n *converges to the corresponding component of* X *in probability* (*almost surely, in mean of order* r) *as* $n \to \infty$. *Depending on the case considered, we then write* $X_n \xrightarrow{p} X$, $X_n \xrightarrow{a.s.} X$ *or* $X_n \xrightarrow{r} X$.

(b) *We say* X_n *converges in law to* X ($X_n \xrightarrow{L} X$) *iff*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \text{ at all continuity points of } F_X(x).$$

where $x = (x_1, x_2, \ldots, x_k)' \in \mathbb{R}^k$,

$$F_{X_n}(x) = P[X_{1n} \le x_1, ..., X_{kn} \le x_k],\ n = 1, 2, ...$$

and

$$F_X(x) = P[X_1 \le x_1, ..., X_k \le x_k].$$

3.2 Theorem UNIVARIATE CHARACTERIZATION OF CONVERGENCE IN LAW FOR A SEQUENCE OF VECTORS.. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of random vectors of dimension* $k \times 1$ *and let* X *be another random vector of dimension* $k \times 1$. *Then*

$$X_n \xrightarrow[n\to\infty]{L} X \Leftrightarrow \lambda' X_n \xrightarrow[n\to\infty]{L} \lambda' X,\ \forall \lambda \in \mathbb{R}^k.$$

In particular, if $X \sim N[\mu, \Sigma]$,

$$X_n \xrightarrow[n\to\infty]{L} N[\mu, \Sigma] \Leftrightarrow \lambda' X_n \xrightarrow[n\to\infty]{L} N[\lambda'\mu, \lambda'\Sigma\lambda], \forall \lambda \in \mathbb{R}^k.$$

3.2. Proofs and additional references

Proofs and further discussions of the results presented above may be found the following references: Rao (1973), Lukacs (1975), Stout (1974), Loève (1977).

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