Jean-Marie Dufour
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TIME SERIES ANALYSIS
EXERCISES
ARIMA PROCESSES

1. Determine which ones among the following processes are stationary (causal) and/or invertible. We suppose

$$(u_t : t \in \mathbb{Z}) \sim WN(0,\ \sigma^2)\ .$$

   (a) $X_t + 0.2\ X_{t-1} - 0.48\ X_{t-2} = u_t$
   (b) $X_t + 1.9\ X_{t-1} + 0.88\ X_{t-2} = u_t + 0.2\ u_{t-1} + 0.7\ u_{t-2}$
   (c) $X_t + 0.6\ X_{t-2} = u_t + 1.2\ u_{t-1}$
   (d) $X_t + 1.8\ X_{t-1} + 0.81\ X_{t-2} = u_t$
   (e) $X_t + 1.6\ X_{t-1} = u_t - 0.4\ u_{t-1} + 0.04\ u_{t-2}$.

2. Consider the model

$$(1 - B + 0.25\ B^2)X_t = (1 + B)u_t$$

   where $(u_t : t \in \mathbb{Z}) \sim WN(0, \sigma^2)$.

   (a) Does this model have a stationary causal solution?
   (b) If so, find
      1. the coefficients of the moving average representation of X_t ;
      2. the autocovariance function of X_t.

   [See Brockwell and Davis (1991, Section 3.3).]

3. Consider the MA(1) model

$$X_t = u_t - \theta u_{t-1}\ ,\ |\theta| < 1\ ,\ t \in \mathbb{Z}$$

   where $(u_t : t \in \mathbf{Z}) \sim WN(0, \sigma^2)$. Derive the partial autocorrelation function of X_t.

4. Consider the MA(1) model

$$X_t = u_t - u_{t-1} \ , \ t \in \mathbb{Z}$$

where $(u_t : t \in \mathbb{Z}) \sim WN(0, \sigma^2)$. Derive the partial autocorrelation function of X_t.

5. Let $(X_t : t \in \mathbb{Z})$ be a stationary (non-causal) model that satisfies the equation

$$X_t = \varphi \, X_{t-1} + u_t \ , \ |\varphi| > 1 \ , \ (u_t : t \in \mathbb{Z}) \sim BB(0, \ \sigma^2) \ .$$

Show that $X_t = (1 \ / \ \varphi)X_{t-1} + \widetilde{u}_t$, $(u_t : t \in \mathbb{Z}) \sim WN(0, \widetilde{\sigma}^2)$ for an appropriately white noise. Determine $\widetilde{\sigma}^2$.

6. Show the recurrence equation

$$X_t = \varphi \, X_{t-1} + u_t \, , \ t \geq 1 \ , \ (u_t : t \in \mathbb{Z}) \sim BB(0, \ \sigma^2)$$

does not have a stationary solution when $|\varphi| = 1$.

7. Let $(Y_t : t \in \mathbb{Z})$ be a stationary second-order process. Show that the recurrence equation

$$X_t - \varphi_1 X_{t-1} - \cdots - \varphi_p X_{t-p} = Y_t - \theta_1 Y_{t-1} - \cdots - \theta_q Y_{t-q}$$

has a stationary solution if $\varphi(z) \equiv 1 - \varphi_1 \, z - ... - \varphi_p \, z^p \neq 0$ for $|z| = 1$. Further, if $\varphi(z) \neq 0$ for $|z| \leq 1$, show that X_t is a causal function of Y_t.

8. Let $(X_t : t \in \mathbb{Z})$ be a stationary ARMA process that satisfies the equation

$$\phi(B) \, X_t = \theta(B) \, u_t \, , \ (u_t : t \in \mathbb{Z}) \sim BB(0, \ \sigma^2)$$

where $\phi(z)$ and $\theta(z)$ are polynomials of finite degree without common root and $\phi(z) \neq 0$ for $|z| = 1$. If $\xi(z)$ is a polynomial, such that $\xi(z) \neq 0$ for $|z| = 1$, show that the equation

$$\xi(B) \, \phi(B) \, Y_t = \xi(B) \, \theta(B) \, u_t$$

has a unique stationary solution $Y_t = X_t$.

9. Consider $(X_t : t \in \mathbb{Z})$ a process such that

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} \ , \ (u_t : t \in \mathbb{Z}) \sim BB(0, \ \sigma^2)$$

where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. Show that $\sum_{j=-\infty}^{\infty} |\gamma(k)| < \infty$ where $\gamma(k)$ is the autocovariance function of X_t.

10. Let $(X_t : t \in \mathbb{Z})$ be a second-order stationary process, and let

$$Y_t = (1 - 0.4B)X_t = X_t - 0.4\, X_{t-1}$$

$$Z_t = (1 - 2.5B)X_t = X_t - 2.5\, X_{t-1}.$$

Show that Y_t and Z_t have the same autocorrelation function.

11. Let $(X_t : t \in \mathbb{Z})$ a second-order stationary ARMA process, such that $\phi(z) \neq 0$ for $|z| \neq 1$, with autocovariance function $\gamma(k)$.

    (a) Show there are constants $c > 0$ and s, where $0 < s < 1$, such that $|\gamma(k)| \leq C\, s^{|k|}$, $k \in \mathbb{Z}$.

    (b) Show that $\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$.

12. Find the coefficients ψ_j , $j = 0,\ 1,\ 2,, \ldots$,of the MA representation

$$X_t = \sum_{j=0}^{\infty} \psi_j\, u_{t-j}$$

of the ARMA$(2, 1)$ process

$$(1 - 0.5B - 0.4B^2)X_t = (1 + 0.25B)\, u_t\ ,\ u_t \sim WN(0, \sigma^2)\ .$$

2. Compute and graph the first ten autocovariances $\gamma(k)$, $k = 1,\ ...\ ,\ 10$, of the process

$$(1 - 0.5B)(1 - 0.4B)(1 - 0.1B)\, X_t = u_t\ ,\ u_t \sim BB(0,\ \sigma^2)\ ,\ t \in \mathbb{Z}\ .$$

3. Compute the mean and the autocovariance function of the process

$$X_t = 2 + 1.3\, X_{t-1} - 0.4\, X_{t-2} + u_t - u_{t-1}\ ,\ u_t \sim BB(0,\ \sigma^2)\ ,\ t \in \mathbb{Z}\ .$$

Is this process causal? Invertible?

4. Let $(X_t : t \in \mathbb{Z})$ an ARMA$(1, 1)$ proces which satisfies the equation

$$X_t - \varphi\, X_{t-1} = u_t + \theta u_{t-1}\ ,\ u_t \sim BB(0,\ \sigma^2)\ ,$$

where $|\varphi| < 1$ and $|\theta| < 1$.

(a) Find the ψ_j coefficients of the MA(∞) representation of X_t .

(b) Show that the aurocorrelation function of X_t is:

$$\begin{aligned} \rho(1) &= (1+\varphi\theta)(\varphi+\theta) \; / \; (1+\theta^2+2\varphi\theta)\,, \\ \rho(k) &= \varphi^{k-1}\rho(1)\;,\; k \geq 1\,. \end{aligned}$$

References

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods*. Springer-Verlag, New York, second edn.