Weak identification in probit models with endogenous covariates[1]

Jean-Marie Dufour[2]

Department of Economics, McGill University, Canada

Joachim Wilde[3]

Department of Business Administration and Economics, Osnabrueck University, Germany

November 2017

ABSTRACT

Weak identification is a well-known issue in the context of linear structural models. However, for probit models with endogenous explanatory variables, this problem has been little explored. In this paper, we study by simulation the behavior of the usual z-test and the LR test in the presence of weak identification. We find that the usual asymptotic z-test exhibits large size distortions (over-rejections under the null hypothesis). The magnitude of the size distortions depends heavily on the parameter value tested. In contrast, asymptotic LR tests do not over-reject and appear to be robust to weak identification.

Keywords: probit model, weak identification

JEL classification: C35

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[1] The authors thank Leandro Magnusson and two anonymous referees for several useful comments, and Sebastian Veldhuis for valuable assistance. This work was supported by the William Dow Chair in Political Economy (McGill University), the Bank of Canada (Research Fellowship), the Toulouse School of Economics (Pierre-de-Fermat Chair of excellence), the Universitad Carlos III de Madrid (Banco Santander de Madrid Chair of excellence), a Guggenheim Fellowship, a Konrad-Adenauer Fellowship (Alexander-von-Humboldt Foundation, Germany), the Natural Sciences and Engineering Research Council of Canada, the Social Sciences and Humanities Research Council of Canada, and the Fonds de recherche sur la société et la culture (Québec).

[2] William Dow Professor of Economics, McGill University, Centre interuniversitaire de recherche en analyse des organisations (CIRANO), and Centre interuniversitaire de recherche en économie quantitative (CIREQ). Mailing address: Department of Economics, McGill University, Leacock Building, Room 414, 855 Sherbrooke Street West, Montreal, Quebec H3A 2T7, Canada. TEL: (1) 514 398 6071; FAX: (1) 514 398 4800;
e-mail: jean-marie.dufour@mcgill.ca. Web page: http://www.jeanmariedufour.com

[3] Corresponding author. Mailing address: Fachbereich Wirtschaftswissenschaften, Rolandstr. 8, 49069 Osnabrueck, Germany; Tel: (49) 541 969 2746; e-mail: Joachim.Wilde@uni-osnabrueck.de.
Web page: https://www.wiwi.uni-osnabrueck.de/en/departments_and_institutes/econometrics_and_statistics_prof_wilde.html

1. Introduction

Probit models are widely used in applied econometrics; for some recent examples, see Abramitzky and Lavy (2014), Beck, Lin and Ma (2014), Bijsterbosch and Dahlhaus (2015), Bouoiyour, Miftah and Mouhoud (2016), Cornelli, Kominek and Ljungqvist (2013), Croushore and Marsten (2016), Engelhardt et al. (2010), Esaka (2010), Fitzenberger, Kohn and Wang (2011), Haider and Jahangir (2017), Hao and Ng (2011), Hlaing and Pourjalali (2012), Horvath and Katuscakova (2016), Khanna, Kim and Lu (2015), Litchfield, Reilly and Veneziani (2012), Massa and Zhang (2013), Wen and Gordon (2014). As in linear models, one or more explanatory variables can be endogenous. This problem can be solved by using instrumental variables; see Wilde (2008) for a comparison of different estimation methods using instrumental variables. The resulting estimates can be used to calculate test statistics for the parameters of the model.

In linear models it is well known that weak instruments may cause considerable size distortions [see Dufour (2003) for an overview]. Wald-type tests like the usual t-tests and F-tests are especially vulnerable to this problem [see Dufour (1997)]. In probit models, a single parameter hypothesis is usually tested by the so-called z-test, i.e. the ratio of a consistent estimate and its asymptotical standard error. This is a Wald-type test. Therefore, big size distortions can be expected. Nevertheless, the topic seems to be largely a white spot in the literature. Exceptions are the recent theoretical papers of Andrews and Cheng (2013, 2014), who address the probit model as an example. However, Andrews and Cheng (2013) restrict their numerical analysis to a probit model with a nonlinear regression function and without endogeneity, and Andrews and Cheng (2014) don't analyze the probit model numerically.[4]

The paper makes several contributions to this problem. First, large size distortions in probit models with endogeneity are demonstrated by a simulation study. Second, we show that size distortions depend heavily on which the parameter value is tested: whereas size distortions are moderate for the problem of testing the null of a zero parameter, testing other values of the parameter yield large size distortions. Third, the behaviour of the classical likelihood ratio statistic in that case is analyzed. For the simulation design considered, no size distortions are observed. However, the probability of type I error can be notably lower than the nominal level of the test (undersizing). Fourth, some new insights concerning the estimation of probit models with endogenous covariates are provided.

Section 2 describes the econometric model and the test statistics. Section 3 explains the simulation design and the estimators used. Since a probit equation is part of the model, some formulae become more complicated than in the linear case. They are described in detail because textbook descriptions are missing so far. Section 4 presents the results of the simulation study, and Section 5 concludes. For ease of exposition, we focus on the binary probit model.

2. Model and classical tests

We study a structural probit model, where one of the explanatory variables is endogenous, and a reduced-form equation for this variable can be specified. The specific model considered is:

$$\begin{aligned} y_{1i}^* &= \gamma_1 y_{2i} + \beta_1 x_{1i} + u_{1i} \\ y_{2i} &= \pi_{21} x_{1i} + \pi_{22} x_{2i} + v_{2i} \end{aligned} \qquad y_{1i} = \begin{cases} 1, & y_{1i}^* > 0 \\ 0, & \text{else} \end{cases}, \quad i = 1, \ldots, N, \tag{2.1}$$

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[4] A further exception is Magnusson (2007), who considered in an early version of his paper the probit model with endogenous covariates as an example and found medium size distortions. However, in later versions of the working paper and in the published version (Magnusson, 2010) the probit example was deleted.

where y_{1i}^* is a latent variable, y_{1i} is its observable indicator, y_{2i} is an endogenous (observable) variable, x_{1i} and x_{2i} are $(K_1\times1)$ and $(K_2\times1)$ vectors of exogenous variables, γ_1, β_1, π_{21}, π_{22} are unknown parameter vectors of dimensions 1, $(1\times K_1)$, $(1\times K_1)$ and $(1\times K_2)$ respectively, and u_{1i} and v_{2i} are error terms with mean zero, variances $\sigma_{u_1}^{\ 2}$ and $\sigma_{v_2}^{\ 2}$ respectively, and $Cov(u_{1i}, u_{1j}) = Cov(v_{2i}, v_{2j}) = Cov(u_{1i}, v_{2j}) = 0$ for $i \neq j$. The probit model assumes that the u_{1i}'s are normally distributed. Whether the distribution of v_{2i} must also be specified depends on the estimation method. If a distribution is needed, we assume that u_{1i} and v_{2i} follow a joint normal distribution, i.e.

$$\left(u_{1i}, v_{2i}\right)\big| x_{1i}, x_{2i} \overset{iid}{\sim} N\left(0, \begin{bmatrix} \sigma_{u_1}^{\ 2} & \sigma_{u_1v_2} \\ \sigma_{u_1v_2} & \sigma_{v_2}^{\ 2} \end{bmatrix}\right).$$

The parameter of special interest is γ_1. It is not identified if π_{22} is equal to zero. Therefore, we say γ_1 that is weakly identified if π_{22} is close to zero. Sometimes weak identification is quantified by the so-called concentration parameter; see Stock, Wright and Yogo (2002, p. 519). However, this parameter grows with N, and hence it suggests that the problem of weak identification is reduced by enlarging the sample size. This is misleading, and therefore we don't use it as guideline in our study.

Testing the significance of γ_1 in empirical studies is usually done by the z-statistic (implemented in almost all econometric software packages):

$$z = \frac{\hat{\gamma}_1}{\sqrt{\hat{V}(\hat{\gamma}_1)}} \tag{2.2}$$

where $\hat{V}(\hat{\gamma}_1)$ is a consistent estimate (assuming identification) of the asymptotic variance of $\sqrt{N}(\hat{\gamma}_1 - \gamma_1)$. If $\hat{\gamma}_1$ is consistent with an asymptotic normal distribution, z is asymptotically standard normal distributed under the assumption of strong identification. The parameter γ_1 can be estimated by two-step methods [see Blundell and Smith (1993) for an overview] or via joint GMM or joint maximum likelihood (ML) estimation of both equations [see Wilde (2008) for a comparison].

The z-test is a Wald-type test. The classical alternatives to it - the Likelihood Ratio (LR) and the Lagrange Multiplier (LM) tests - are based on ML estimation of the parameters. In linear models, the latter are less affected by weak identification than Wald-type tests. Here, we focus on the LR test. Given the estimates, the LR-statistic is calculated easily, whereas for the LM test an estimation of the complicated information matrix is needed, and the results may depend on which estimation procedure was chosen. The loglikelihood function of model (2.1) under the standard assumptions above is:

$$\begin{aligned} \ln l(\theta) = \sum_{i=1}^{N} \Bigg[& -0.5\ln\left(2\Pi\sigma_{v_2}^{\ 2}\right) - 0.5\left(\frac{y_{2i} - \pi_2 x_i}{\sigma_{v_2}}\right)^2 \\ & + y_{1i} \ln\Phi\left(\frac{1}{\sqrt{1-\rho_v^{\ 2}}}\left(\frac{(\gamma_1\pi_{21} + \beta_1)x_{1i} + \gamma_1\pi_{22}x_{2i}}{\sigma_{v_1}} + \rho_v\left(\frac{y_{2i} - \pi_2 x_i}{\sigma_{v_2}}\right)\right)\right) \\ & + (1 - y_{1i})\ln\left(1 - \Phi\left(\frac{1}{\sqrt{1-\rho_v^{\ 2}}}\left(\frac{(\gamma_1\pi_{21} + \beta_1)x_{1i} + \gamma_1\pi_{22}x_{2i}}{\sigma_{v_1}} + \rho_v\left(\frac{y_{2i} - \pi_2 x_i}{\sigma_{v_2}}\right)\right)\right)\right)\Bigg], \end{aligned} \tag{2.3}$$

where $x_i = (x_{1i}', x_{2i}')'$, $v_{1i} = u_{1i} + \gamma_1 v_{2i}$, $\theta = (\gamma_1, \beta_1, \pi_2, \sigma_{v_2}, \rho_v)'$, $\pi_2 = (\pi_{21}, \pi_{22})$, $\sigma_{v_1} = \sqrt{\mathrm{Var}(v_{1i})}$, and $\rho_v = \mathrm{Corr}(v_{1i}, v_{2i})$; see Wilde (2008, appendix 2). Since the structural parameters enter the likelihood only through ratios with a standard deviation and the latter does not appear separately, only these ratios are identifiable. Therefore, in our simulation study, σ_{v_1} is taken as known.

We consider the problem of testing

$$H_0: \gamma_1 = \tilde{\gamma}_1 \quad \text{vs.} \quad H_1: \gamma_1 \neq \tilde{\gamma}_1 .$$

We denote by $\hat{\theta}_{ML}$ the unrestricted ML estimator of θ (based on (2.3)) and by $\hat{\theta}_{RML}$ the restricted ML estimator under the null hypothesis. The LR statistic takes the form

$$LR = 2\left(\ln l\left(\hat{\theta}_{ML}\right) - \ln l\left(\hat{\theta}_{RML}\right)\right).$$

Under the usual assumptions (including strong identification) LR is asymptotically $\chi^2(1)$ distributed.

3. Simulation design

In order to avoid arbitrary choices of unnecessary nuisance parameters, we consider a simple simulation design with $K_1 \in \{0, 1\}$ and $K_2 = 1$. $K_1 = 0$ defines a model even without constants, whereas $K_1 = 1$ is used to add a constant in both equations. With $K_1 = 0$, ML estimation causes numerical problems, i.e. many replications ended with the message "maximum iterations reached" even after 200 and more iterations. Since GMM estimation is a well-known alternative to ML estimation also for probit type models and the efficiency loss in models like ours seems to be small [see Wilde (2008)], we used GMM estimation for the z-test. As will be shown below, the results under weak identification are nearly the same for models with and without constants. Therefore, we focus our comparison of z-test and LR test for models including constants in both equations.

In (2.1), the second equation is a reduced-form equation. Endogeneity of y_2 can occur for at least two reasons: correlation between the error terms of the structural equations and/ or simultaneity between y_1 and y_2. Both yield a correlation between u_1 and v_2. In our simulation, we focus on simultaneity because it can be interpreted more easily. Nevertheless, all results can be reproduced by assuming correlation between the error terms of the structural equations. Therefore, our main data generating model including constants is:

$$\begin{aligned} y_{1i}^* &= \gamma_1 y_{2i} + \beta_{11} + u_{1i} \\ y_{2i} &= \gamma_2 y_{1i}^* + \beta_{21} + \beta_{22} x_i + u_{2i} \end{aligned} \qquad y_{1i} = \begin{cases} 1, & y_{1i}^* > 0 \\ 0, & \text{elsewhere} \end{cases} \qquad i = 1, ..., N \tag{3.1}$$

The residuals u_{1i} and u_{2i} are drawn independently from a N(0, 16) distribution, i.e. the residual variances are equal for both equations. The exogenous variable x_i is drawn from a N(0.5, 16) distribution, so the expected number of ones for y_{1i} differs from the expected number of zeros for the model without constants. Alternatively, we draw the residuals independently from a N(0, 1) distribution and x_i from a N(0.5, 1) distribution. The constants are chosen as $\beta_{11} = 0.5$ and $\beta_{21} = 0.25$. Weak identification is equivalent to β_{22} close to zero. In our simulation, we choose $\beta_{22} = 0.0001$. Smaller values of β_{22} do not sharpen the results any more. The case of strong identification is simulated by $\beta_{22} = 1$. Furthermore, all simulations are also done for $\beta_{22} = 0.1$ to see whether weak identification causes problems also for moderate parameter values. The simulations are done for the sample sizes N = 400 (medium sample size) and N = 2000 (large sample), and are replicated 5000 times.

The estimated model is:

$$\begin{aligned} y_{1i}^{*} &= \gamma_1 y_{2i} + \beta_{11} + u_{1i} \\ y_{2i} &= \pi_{21} + \pi_{22} x_i + v_{2i} \end{aligned} \qquad y_{1i} \text{ as defined in (3.1)}, \quad i = 1, ..., N, \tag{3.2}$$

where $\pi_{21} = \dfrac{\gamma_2\beta_{11} + \beta_{21}}{1-\gamma_1\gamma_2}$, $\pi_{22} = \dfrac{\beta_{22}}{1-\gamma_1\gamma_2}$, and $v_{2i} = \dfrac{\gamma_2 u_{1i} + u_{2i}}{1-\gamma_1\gamma_2}$. In (3.2), γ_1 is exactly identified as long as π_{22} is different from zero.

(3.2) is estimated by GMM using the "natural" moment conditions [see Wilde (2008)]:

$$E\begin{bmatrix} x_i\left(y_{1i} - \Phi\left(\left(\gamma_1\left(\pi_{21}+\pi_{22}x_i\right)+\beta_{11}\right)/\sigma_{v_1}\right)\right) \\ x_i\left(y_{2i} - \pi_{21} - \pi_{22}x_i\right) \\ 1\left(y_{1i} - \Phi\left(\left(\gamma_1\left(\pi_{21}+\pi_{22}x_i\right)+\beta_{11}\right)/\sigma_{v_1}\right)\right) \\ 1\left(y_{2i} - \pi_{21} - \pi_{22}x_i\right) \end{bmatrix} = 0 .$$

Setting

$$\theta = \begin{pmatrix} \gamma_1 \\ \pi_{22} \\ \beta_{11} \\ \pi_{21} \end{pmatrix}, \; m_i(\theta) = \begin{pmatrix} x_i\left(y_{1i} - \Phi\left(\left(\gamma_1\left(\pi_{21}+\pi_{22}x_i\right)+\beta_{11}\right)/\sigma_{v_1}\right)\right) \\ x_i\left(y_{2i} - \pi_{21} - \pi_{22}x_i\right) \\ 1\left(y_{1i} - \Phi\left(\left(\gamma_1\left(\pi_{21}+\pi_{22}x_i\right)+\beta_{11}\right)/\sigma_{v_1}\right)\right) \\ 1\left(y_{2i} - \pi_{21} - \pi_{22}x_i\right) \end{pmatrix}, \; \bar{m}(\theta) = \frac{1}{N}\sum_{i=1}^{N} m_i(\theta),$$

we calculated

$$\hat{\theta}_{GMM} = \arg\min_{\theta}\left\{\bar{m}(\theta)' W_N \bar{m}(\theta)\right\}, \; W_N \text{ a weighting matrix,}$$

with regard to θ. Since the number of moment conditions is equal to the number of parameters the weighting matrix in the criterion function of the GMM estimator does not matter theoretically and the same asymptotic covariance matrix of the estimator can be used for all choices of W_N [Harris and Mátyás (1999, p. 22)]. To be more precise, the asymptotic covariance matrix is [see Greene (2008, p. 445)]:

$$\text{asyVar}\left(\hat{\theta}\right) = \frac{1}{N}\left[G'\Psi^{-1}G\right]^{-1},$$

$$\Psi = \text{asyVar}\left(\sqrt{N}\bar{m}\right), \; \varphi_i := \varphi\left(\left(\gamma_1\left(\pi_{21}+\pi_{22}x_i\right)+\beta_{11}\right)/\sigma_{v_1}\right),$$

$$G = \frac{\partial \bar{m}}{\partial \theta'} = \begin{pmatrix} \frac{1}{N}\sum_{i=1}^{N} -\frac{\pi_{21}x_i + \pi_{22}x_i^2}{\sigma_{v_1}}\varphi_i & \frac{1}{N}\sum_{i=1}^{N} -\frac{\gamma_1 x_i^2}{\sigma_{v_1}}\varphi_i & \frac{1}{N}\sum_{i=1}^{N} -\frac{x_i}{\sigma_{v_1}}\varphi_i & \frac{1}{N}\sum_{i=1}^{N} -\frac{\gamma_1 x_i}{\sigma_{v_1}}\varphi_i \\ 0 & \frac{1}{N}\sum_{i=1}^{N} -x_i^2 & 0 & \frac{1}{N}\sum_{i=1}^{N} -x_i \\ \frac{1}{N}\sum_{i=1}^{N} -\frac{\pi_{21} + \pi_{22}x_i}{\sigma_{v_1}}\varphi_i & \frac{1}{N}\sum_{i=1}^{N} -\frac{\gamma_1 x_i}{\sigma_{v_1}}\varphi_i & \frac{1}{N}\sum_{i=1}^{N} -\frac{1}{\sigma_{v_1}}\varphi_i & \frac{1}{N}\sum_{i=1}^{N} -\frac{\gamma_1}{\sigma_{v_1}}\varphi_i \\ 0 & \frac{1}{N}\sum_{i=1}^{N} -x_i & 0 & -1 \end{pmatrix}$$

Given the assumptions above this matrix can be estimated consistently by

$$\text{est asyVar}\left(\hat{\theta}\right) = \frac{1}{N}\left[\hat{G}'\hat{\Psi}^{-1}\hat{G}\right]^{-1},$$

$$\hat{\Psi} = \frac{1}{N}\sum_{i=1}^{N} m_i\left(\hat{\theta}\right)m_i\left(\hat{\theta}\right)', \ \hat{G} = G \text{ after substituting } \hat{\theta} \text{ for } \theta.$$

The square root of the first diagonal element is the denominator of the z-statistic (2.2). $\hat{\Psi}^{-1}$ is the optimal weighting matrix and is used to calculate the nominator of (2.2).[5] In the model without constants only the first two moment conditions are used (setting π_{21} and β_{11} equal to zero), and G reduces to a 2×2 matrix.

Concerning the ML estimation of the model without constants everything works fine for strong identification, whereas for weak identification the algorithm did not find the maximum for nearly half of the replications. This is caused by the following: In the model without constants, for instance the second summand of (2.3) reduces to

$$y_{1i} \ln \Phi\left(\frac{1}{\sqrt{1-\rho_v^{\,2}}}\left(\frac{\gamma_1 \pi_{22} x_i}{\sigma_{v_1}} + \rho_v\left(\frac{y_{2i} - \pi_{22} x_i}{\sigma_{v_2}}\right)\right)\right).$$

Thus, the parameter γ_1 is part of the log-likelihood function only in the product $\gamma_1\pi_2$. However, in case of weak identification, π_2 is close to zero, so that it is very difficult to find the "true" value of γ_1. The log-likelihood function is rather flat concerning γ_1. However, including constants avoids the numerical problems discussed above without changing the results concerning weak identification. The latter aspect was confirmed for those parameter values for which the optimum was also found in the model without constants.[6]

4. Simulation results

We distinguish two cases. First, we test the null hypothesis $\gamma_1 = 0$, i.e. the significance of y_2. Second, we test the null hypothesis $\gamma_1 = c$, c a constant different from zero, and present the results for $c = 2$. In both cases, the simulations are done for different values of γ_2. Small values of γ_2 correspond with a "small" problem of simultaneity. We choose $\gamma_2 = \pm 0.5$. A "medium" problem of simultaneity is represented by $\gamma_2 = \pm 1.5$, and a "large" problem of simultaneity is represented by $\gamma_2 = \pm 3$ and $\gamma_2 = \pm 6$.

4.1 Results for the z-test

a) Main model

The results for the models with and without constants are rather similar. Therefore, we focus on the model with constants. The results for the model without constants can be found in the appendix. In case of strong identification and testing significance, no size problems can be observed for a nominal size of 10% and 5% and mostly for a nominal size of 1% (see Table 1). In case of weak identification the picture is mixed.[7] The key result is highlighted in Figure 1 for N = 2000 and a nominal size of 5%. If there is only "weak" simultaneity, an extreme undersizing is observed.

[5] All simulations were done using R. The GMM estimation was done using the package "GMM", version 1.6-1. The case of iid observations can be implemented by the option vcov = "iid"; see Chaussé (2010, p.13). All R codes are available on request.

[6] For the ML estimation the exogenous variable x_i was drawn from a N(0, 16) distribution.

[7] This result is similar to that of Magnusson (2007).

However, "strong" simultaneity causes medium size distortions. Nevertheless, the size distortions are smaller than those in linear simultaneous equations. This is surprising because weak identification should cause similar problems in probit models. Furthermore, the results for $\beta_{22} = 0.1$ are close to the results for $\beta_{22} = 0.0001$, i.e. oversizing occurs even for moderate values of the parameter β_{22}. All results are similar for N = 2000 and N = 400.

Next we consider the results concerning data simulated with $\gamma_1 = 2$ and testing H_0: $\gamma_1 = 2$. Again, in case of strong identification, no size distortion appears for a nominal size of 10% and 5%; see Table 2. However, in case of weak identification and strong simultaneity, the size distortions become very large ($\gamma_2 = 0.5$ has to be omitted here because γ_1 is no longer identified in that case); see Figure 2. The empirical size becomes more than tenfold as high as the nominal size. Thus, in probit models similar size distortions as in linear simultaneous equations models can be observed. In case of strong simultaneity size distortions also occur for $\beta_{22} = 0.1$. They are stronger for N = 400, i.e. sample size matters in that case.

The differences between testing $\gamma_1 = 0$ and $\gamma_1 = 2$ demonstrate an important feature: in probit models, the size distortion depends heavily on the parameter value tested. This property is not easily explained by the concentration parameter. Following the definition of Stock, Wright and Yogo (2002, p. 519), in our model without constants the concentration parameter μ^2 is:

$$\mu^2 = \left(\sum_{i=1}^{N} x_i^2 \pi_{22}^2\right) \Big/ \sigma_{v_2}^2 = \left(\sum_{i=1}^{N} x_i^2 \left(\frac{\beta_{22}}{1-\gamma_1\gamma_2}\right)^2\right) \Big/ \left(\frac{\sigma_{u_2}^2 + \gamma_2^2 \sigma_{u_1}^2}{(1-\gamma_1\gamma_2)^2}\right) = \left(\sum_{i=1}^{N} x_i^2 \beta_{22}^2\right) \Big/ \left(\sigma_{u_2}^2 + \gamma_2^2 \sigma_{u_1}^2\right).$$

The concentration parameter does not depend on γ_1. In the model with constants the formula is more bulky, but the message remains unchanged. Thus, concerning the concentration parameter the problem of weak identification should not depend on the value of γ_1.

b) Sensitivity analysis

We do several checks of robustness of our results in Section 4.1a. First, we vary the estimation method, i.e. we iterate our two-step GMM estimator, and we use I instead of $\hat{\Psi}^{-1}$ as weighting matrix.[8] Second, we changed the variances, i.e. we draw the u_i's from a N(0, 1) distribution and x_i from a N(0.5, 1) distribution. Third, we change from exact identification of γ_1 to underidentification, i.e. $\beta_{22} = 0$, and overidentification, i.e. we add a second exogenous variable in the second equation. Except overidentification all variations do not change the results. Thus, we present only examples of the results for these variations.

Varying the estimation method does not change the results, i.e. the shares of rejections are exactly the same as in tables 1 and 2. This is plausible because in case of exact identification all unbiased estimation methods should lead to the same results. Varying the variances in the sample design changes the results slightly, but the tendency remains still the same: Only moderate size distortions testing $\gamma_1 = 0$ and large size distortions testing $\gamma_1 = 2$ are observed (see Table 3). The results in case of underidentification are very close to those of $\beta_2 = 0.0001$, i.e. underidentification does not sharpen the results anymore (see Table 4).

In contrast with the results above, overidentification leads to new insights. In the second equations of (3.1) and (3.2), there is an additional exogenous variable drawn from a N(0, 16) distribution. The corresponding structural parameter β_{23} varies in the same manner as the parameter β_{22}, i.e. in case of strong (over-)identification we choose $\beta_{22} = \beta_{23} = 1$, in case of weak (over-)identification $\beta_{22} = \beta_{23} = 0.0001$, and $\beta_{22} = \beta_{23} = 0.1$ again represents a moderate value of the parameters. The second exogenous variable is used as an instrument in the same way as x_i and the

[8] We also tried the option "CUE" for the continuous updating estimator. However, we always got the error message "node stack overflow".

ones in the simulations before. The matrix G now becomes a (6×5)-matrix, i.e. the number of moment conditions exceeds the number of parameters. Thus, the choice of the weighting matrix should matter. Therefore, in the following we contrast the results using the identity matrix I with those based on the "optimal" weighting matrix. We focus on a nominal size of $\alpha = 5\%$.

First, sample size now also matters. Whereas in case of strong (over-)identification everything works fine for N = 2000 (see Table 5), N = 400 is too small to meet the nominal size. For testing $\gamma_1 = 0$ the share of rejections is in the interval [0.357, 0.3754]; for testing $\gamma_1 = 2$ the share of rejections is in the interval [0.2846, 0.3942]. This is an interesting side result and a cautious note if models like ours are used with macroeconomic or experimental data, where even N = 400 is a challenging sample size.

Second, for testing $\gamma_1 = 0$ with N = 2000 leads to stronger size distortions in case of weak identification and large simultaneity. Now, the size distortion is more than fourfold as high as the nominal size (see Table 5). The results are slightly stronger if I is used as weighting matrix. Again, in case of weak identification and weak simultaneity undersizing is observed although the undersizing now becomes smaller. In case of strong simultaneity the results for $\beta_{22} = \beta_{23} = 0.1$ are again close to weak identification, whereas in case of weak simultaneity the results for $\beta_{22} = \beta_{23} = 0.1$ are close to strong identification.

Third, when testing $\gamma_1 = 2$ with N = 2000, the picture is mixed. Using weighting matrix I strengthens the results in case of strong simultaneity in comparison with the results for model (3.2). The results are now close to the theoretical expectation that the probability of rejections grows up to one. However, using the optimal weighting matrix gives lower size distortions than in case of exact identification. The interpretation of this result is difficult: for some parameter combinations, the shares of rejections for $\beta_{22} = \beta_{23} = 0.0001$ are even lower than for $\beta_{22} = \beta_{23} = 0.1$. This may be due to the fact that the asymptotic standard errors are a complicated function of γ_1. If the identity matrix I is used as the weighting matrix, only $G'G$ needs to be inverted for calculating the asymptotic variance-covariance matrix [see Greene (2008, p. 451)]. In contrast, using the optimal weighting matrix requires the inversion of $G'\Psi^{-1}G$. In case of weak identification the latter calculation may lead to "bad" results. However, further research is needed to clarify the reasons for this result. Nevertheless, in case of strong and even medium simultaneity always noteworthy size distortions are observed.

4.2 Results for the LR test for the main model

We consider again our main model (3.2). We use the same parameter values as in Section 4.1a and calculate the Maximum Likelihood estimator and the LR statistic. In case of strong identification, again no size problems can be observed (see Table 6). However, the results under weak identification differ substantially from those for the z-test. If simultaneity is weak, the observed share of rejections is near to the true size, if it is medium or strong, *under*sizing is observed. Thus, the LR test may be a conservative alternative to the z-test.

Testing significance, results for $\gamma_2 = \pm 6$ are missing, because the program stopped with an error message for some replications. This message was caused by the following: Consider for instance $\gamma_1 = 0$ and $\gamma_2 = 6$. This implies $\rho_v = 0.9864$, i.e. the bivariate normal distribution of v_1 and v_2 is near to singularity. Furthermore, in (2.3) $1/\sqrt{1-\rho_v^2} = 6.08$, $\Phi(\approx 6) = 1$, $1-\Phi(\approx 6) = 0$, and $\ln(1-\Phi(\approx 6))$ is not defined. Therefore, ML estimation is less robust against a high correlation of the reduced-form errors than GMM estimation. This is similar to the findings in Wilde (2008, p. 476) and an interesting side result of our paper.

In contrast with the z-test, the results for the LR test under weak identification do *not* change if $\gamma_1 = 2$ is tested. Furthermore, in that case $\gamma_2 = 6$ is possible. With $\beta_{22} = 0.1$ and N = 2000, only very strong simultaneity ($\gamma_2 = \pm 6$) leads to undersizing if N = 2000, i.e. the results are more stable für moderate values of β_{22} than those in case of the z-test.

5. Conclusion

The paper analyses weak identification in probit models with endogenous covariates. It shows remarkable size distortions concerning the usual z-test. However, further research is needed to clarify why the magnitude depends heavily on the parameter value tested. The likelihood ratio statistic seems to be a conservative alternative which is robust to weak identification. Further research is useful to clarify how advanced methods like those of Andrews and Cheng (2014), Dufour (2006) or Kleibergen (2005) will work for probit models with endogenous covariates.

Table 1: Rejection frequencies of the z-test, H$_0$: $\gamma_1 = 0$

		$\beta_{22} = 1$			$\beta_{22} = 0.1$			$\beta_{22} = 0.0001$		
γ_2	Nominal level	10%	5%	1%	10%	5%	1%	10%	5%	1%
−6	N = 2000	0.0908	0.0484	0.0178	0.1634	0.1142	0.0512	0.169	0.1114	0.0464
	N = 400	0.0984	0.0688	0.032	0.1764	0.1206	0.05	0.17	0.1196	0.0514
−3	N = 2000	0.095	0.0502	0.0118	0.131	0.088	0.0414	0.1424	0.0916	0.035
	N = 400	0.0868	0.0508	0.0206	0.1528	0.1018	0.0394	0.1452	0.0938	0.0382
−1.5	N = 2000	0.0988	0.0506	0.0114	0.0942	0.0626	0.0242	0.0854	0.0508	0.0124
	N = 400	0.0964	0.0482	0.0136	0.1044	0.064	0.0196	0.0896	0.0528	0.0144
−0.5	N = 2000	0.0984	0.05	0.0102	0.07	0.034	0.0058	0.012	0.003	0
	N = 400	0.0996	0.051	0.0116	0.0432	0.0202	0.0012	0.0148	0.0056	0.0006
0.5	N = 2000	0.0986	0.047	0.011	0.0698	0.032	0.008	0.0128	0.0046	0.0004
	N = 400	0.0974	0.05	0.0102	0.0382	0.0154	0.0016	0.0126	0.0038	0.0006
1.5	N = 2000	0.097	0.0486	0.01	0.0914	0.0626	0.0254	0.0836	0.0458	0.0136
	N = 400	0.0966	0.048	0.0102	0.096	0.0566	0.0178	0.0896	0.048	0.0128
3	N = 2000	0.096	0.0468	0.0108	0.1212	0.0866	0.0388	0.143	0.0922	0.0342
	N = 400	0.0864	0.0464	0.016	0.1408	0.0914	0.0332	0.1488	0.0976	0.034
6	N = 2000	0.0886	0.0462	0.016	0.1548	0.1076	0.0484	0.1674	0.1138	0.0458
	N = 400	0.093	0.0658	0.0276	0.1696	0.113	0.0454	0.1768	0.1196	0.0482

Figure 1: Rejection frequencies of the z-test under weak identification,
H$_0$: $\gamma_1 = 0$, nominal size 5%, N = 2000

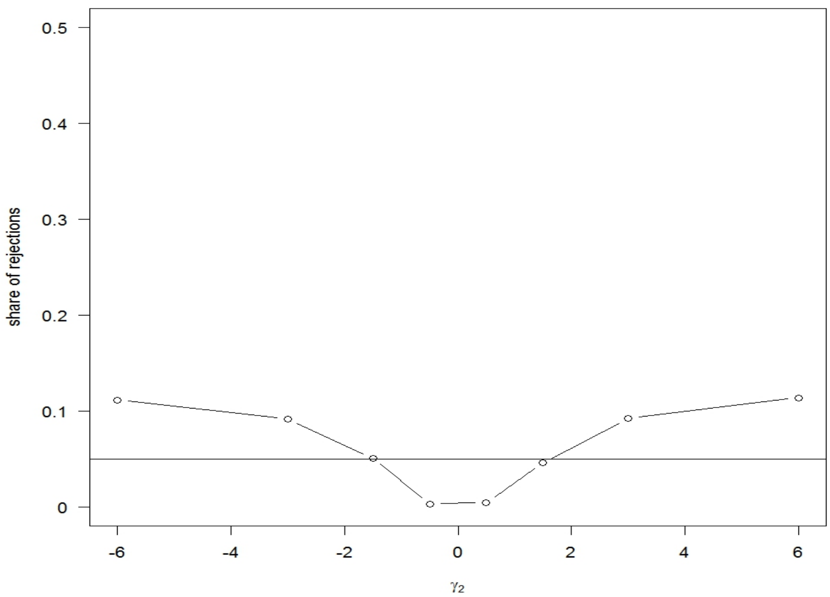

Table 2: Rejection frequencies of the z-test, H_0: $\gamma_1 = 2$

		$\beta_{22} = 1$			$\beta_{22} = 0.1$			$\beta_{22} = 0.0001$		
γ_2	Nominal level	10%	5%	1%	10%	5%	1%	10%	5%	1%
−6	N = 2000	0.089	0.0556	0.0234	0.2724	0.228	0.1696	0.5586	0.5142	0.4366
	N=400	0.1038	0.0726	0.0382	0.448	0.4078	0.3322	0.5732	0.526	0.454
−3	N=2000	0.0996	0.0518	0.0144	0.1586	0.1244	0.081	0.3686	0.31	0.2076
	N=400	0.0842	0.048	0.02	0.2772	0.2314	0.1634	0.3756	0.315	0.218
−1.5	N=2000	0.097	0.0498	0.0108	0.1042	0.0756	0.032	0.1562	0.0996	0.0404
	N=400	0.0932	0.049	0.0116	0.146	0.1002	0.0434	0.1632	0.104	0.0432
−0.5	N=2000	0.099	0.0468	0.0108	0.0684	0.0316	0.0056	0.011	0.0026	0
	N=400	0.0928	0.049	0.0086	0.0418	0.015	0.0012	0.0126	0.0032	0.0002
1.5	N=2000	0.0996	0.0494	0.0102	0.0864	0.054	0.0218	0.0694	0.0358	0.0078
	N=400	0.0914	0.0424	0.0084	0.0856	0.0522	0.0166	0.07	0.04	0.0084
3	N=2000	0.0946	0.0472	0.0116	0.1576	0.1182	0.0708	0.3184	0.253	0.1554
	N=400	0.0774	0.0426	0.0146	0.2496	0.1968	0.1246	0.3246	0.2608	0.162
6	N=2000	0.0872	0.0482	0.0198	0.264	0.226	0.1674	0.5416	0.4966	0.4124
	N=400	0.1036	0.0696	0.0334	0.4446	0.4004	0.3256	0.5592	0.5152	0.4304

Figure 2: Rejection frequencies of the z-test under weak identification, H_0: $\gamma_1 = 2$, nominal size 5%, N=2000

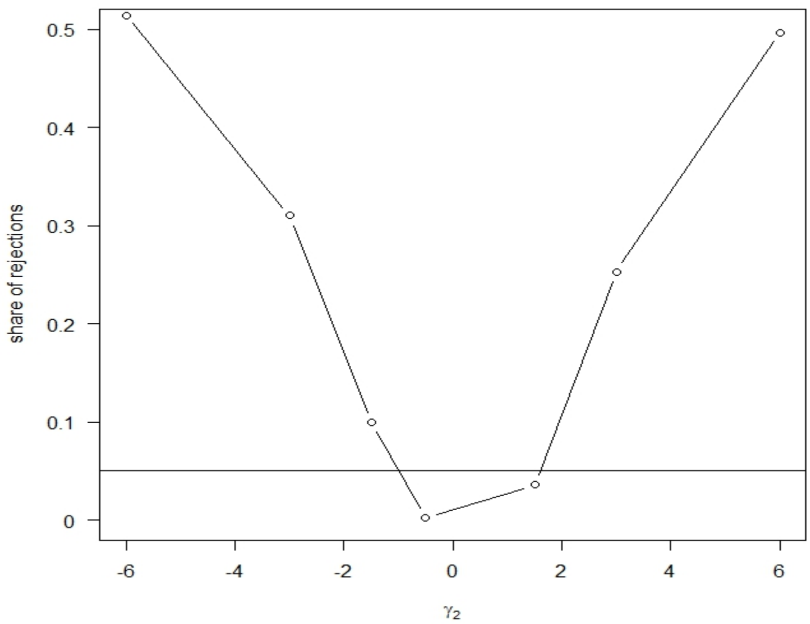

Table 3: Rejection frequencies of the z-test, sample design with variances = 1

			$\beta_{22} = 1$			$\beta_{22} = 0.1$			$\beta_{22} = 0.0001$		
γ_1	γ_2	Nominal level	10%	5%	1%	10%	5%	1%	10%	5%	1%
0	6	N = 2000	0.0908	0.0474	0.0162	0.1278	0.0834	0.0352	0.1424	0.0898	0.033
		N = 400	0.0954	0.0618	0.0252	0.1358	0.0852	0.034	0.148	0.0946	0.0332
2	6	N = 2000	0.0808	0.051	0.0176	0.3126	0.2448	0.1708	0.5718	0.5084	0.4078
		N = 400	0.1054	0.0724	0.0352	0.4376	0.3902	0.311	0.5464	0.5034	0.4194

Table 4: Rejection frequencies of the z-test in case of underidentification

			$\beta_{22} = 0$	
γ_2	Nominal level	*10%*	*5%*	*1%*
6	N = 2000	0.1674	0.1138	0.0458
	N = 400	0.177	0.1194	0.0482
6	N = 2000	0.5424	0.4964	0.4116
	N = 400	0.5592	0.515	0.430

Table 5: Rejection frequencies of the z-test, nominal level 5%, N = 2000, with overidentification of γ_1 in the theoretical model

		$\beta_{22} = \beta_{23} = 1$		$\beta_{22} = \beta_{23} = 0.1$		$\beta_{22} = \beta_{23} = 0.0001$	
		Weighting matrix		Weighting matrix		Weighting matrix	
γ_1	γ_2	I	Optimal	I	Optimal	I	optimal
0	−6	0.0526	0.045	0.2396	0.1724	0.2682	0.2072
0	−3	0.0484	0.0464	0.1578	0.11	0.2304	0.1758
0	−1.5	0.0466	0.0464	0.0818	0.0686	0.135	0.1174
0	−0.5	0.0466	0.0464	0.0456	0.0488	0.0112	0.0122
0	0.5	0.047	0.0456	0.0458	0.0456	0.0124	0.015
0	1.5	0.0474	0.046	0.0826	0.071	0.131	0.1232
0	3	0.0492	0.0464	0.1482	0.1062	0.2252	0.1904
0	6	0.0548	0.0496	0.2256	0.1692	0.2686	0.212
2	−6	0.0376	0.0454	0.358	0.1574	0.7844	0.099
2	−3	0.0452	0.0486	0.1952	0.113	0.5708	0.1126
2	−1.5	0.047	0.0456	0.0888	0.0704	0.2416	0.1156
2	−0.5	0.0442	0.0458	0.044	0.0446	0.0108	0.0154
2	1.5	0.0538	0.0548	0.0714	0.0598	0.11	0.1104
2	3	0.046	0.0504	0.2006	0.1106	0.4968	0.2102
2	6	0.047	0.0542	0.3882	0.1606	0.7736	0.1344

Table 6: Rejection frequencies of the LR test

			$\beta_{22} = 1$			$\beta_{22} = 0.1$			$\beta_{22} = 0.0001$		
γ_1	γ_2	nominal size	10%	5%	1%	10%	5%	1%	10%	5%	1%
0	−3	N = 2000	0.101	0.0494	0.01	0.096	0.0464	0.0082	0.0654	0.0283	0.0048
		N = 400	0.0992	0.0518	0.0128	0.0798	0.0388	0.009	0.0698	0.0324	0.0066
0	−1.5	N = 2000	0.1	0.0492	0.0106	0.0998	0.049	0.0104	0.0828	0.0366	0.0062
		N = 400	0.1	0.0514	0.0118	0.0946	0.0476	0.0108	0.0834	0.0408	0.0072
0	−0.5	N = 2000	0.0986	0.0488	0.012	0.0986	0.0488	0.012	0.0958	0.0478	0.0108
		N = 400	0.0996	0.0522	0.0112	0.0996	0.052	0.011	0.0956	0.051	0.011
0	0.5	N = 2000	0.1	0.0488	0.0112	0.1	0.0488	0.0112	0.0966	0.0484	0.0098
		N = 400	0.0996	0.0534	0.0116	0.0992	0.053	0.0114	0.0962	0.0518	0.011
0	1.5	N = 2000	0.0988	0.0538	0.011	0.0986	0.0536	0.0106	0.081	0.041	0.008
		N = 400	0.099	0.051	0.0106	0.0948	0.0434	0.0086	0.0806	0.0414	0.007
0	3	N = 2000	0.0998	0.0548	0.0106	0.0956	0.0488	0.0076	0.0664	0.034	0.0054
		N = 400	0.0962	0.0496	0.0102	0.0732	0.0324	0.0042	0.0676	0.0298	0.005
2	−6	N = 2000	0.1014	0.0536	0.01	0.0648	0.0312	0.0066	0.051	0.024	0.0042
		N = 400	0.0956	0.0514	0.0114	0.0482	0.0244	0.0058	0.0456	0.0238	0.0044
2	−3	N = 2000	0.0974	0.052	0.0092	0.0942	0.0482	0.0092	0.0554	0.024	0.0046
		N = 400	0.099	0.0504	0.0106	0.062	0.03	0.0068	0.0498	0.0248	0.005
2	−1.5	N = 2000	0.094	0.0466	0.0096	0.0978	0.0514	0.0114	0.074	0.035	0.0056
		N = 400	0.0972	0.0524	0.0078	0.0826	0.0416	0.007	0.0682	0.0328	0.0066
2	−0.5	N = 2000	0.0974	0.0478	0.0104	0.0968	0.0478	0.0124	0.0936	0.0472	0.0102
		N = 400	0.101	0.0462	0.0078	0.0954	0.0486	0.0086	0.095	0.0456	0.0092
2	1.5	N = 2000	0.099	0.0514	0.01	0.0994	0.0522	0.0096	0.0808	0.0404	0.0084
		N = 400	0.0922	0.0496	0.0092	0.0914	0.0474	0.0062	0.0814	0.038	0.0072
2	3	N = 2000	0.0996	0.0492	0.0104	0.0926	0.0454	0.0062	0.0588	0.0284	0.0056
		N = 400	0.094	0.0476	0.0074	0.0708	0.0302	0.002	0.0538	0.0258	0.0046
2	6	N = 2000	0.0994	0.0492	0.0106	0.0678	0.0276	0.0046	0.0492	0.0268	0.0062
		N = 400	0.0894	0.0462	0.0084	0.0492	0.0218	0.0036	0.0462	0.0238	0.0052

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Appendix

Simulation results for model (3.2) with $\beta_{11} = \pi_{21} = 0$

Table A1: Rejection frequencies of the z-test, H_0: $\gamma_1 = 0$.

		$\beta_{22} = 1$			$\beta_{22} = 0.1$			$\beta_{22} = 0.0001$		
γ_2	Nominal size	10%	5%	1%	10%	5%	1%	10%	5%	1%
−6	N = 2000	0.0888	0.0482	0.0178	0.1586	0.1118	0.0516	0.1724	0.1172	0.0474
	N = 400	0.0982	0.0668	0.0308	0.1774	0.1188	0.0544	0.1776	0.1232	0.0498
−3	N = 2000	0.0982	0.047	0.012	0.1248	0.0894	0.0408	0.1508	0.0942	0.035
	N = 400	0.0866	0.0478	0.019	0.1516	0.1012	0.0396	0.1534	0.101	0.035
−1.5	N = 2000	0.0998	0.0484	0.009	0.093	0.0636	0.0242	0.0892	0.049	0.014
	N = 400	0.096	0.0468	0.0126	0.1042	0.0608	0.0214	0.094	0.0528	0.0132
−0.5	N = 2000	0.1002	0.0502	0.0098	0.0714	0.034	0.0078	0.0128	0.003	0
	N = 400	0.0958	0.047	0.011	0.0456	0.0188	0.0018	0.0144	0.005	0.0002
0.5	N = 2000	0.1018	0.051	0.0074	0.0712	0.033	0.0096	0.0116	0.0042	0.0002
	N = 400	0.0948	0.0478	0.0096	0.0398	0.0172	0.0026	0.0132	0.0046	0.0004
1.5	N = 2000	0.0992	0.0492	0.0088	0.0956	0.0618	0.0242	0.086	0.0488	0.0132
	N = 400	0.0916	0.0482	0.01	0.0972	0.0602	0.0194	0.0904	0.0508	0.0124
3	N = 2000	0.0964	0.0492	0.0094	0.1264	0.0888	0.0392	0.1494	0.0942	0.0342
	N = 400	0.0862	0.0464	0.0152	0.1438	0.0956	0.037	0.1556	0.1004	0.036
6	N = 2000	0.0902	0.0482	0.0136	0.1596	0.1108	0.0504	0.1736	0.1192	0.0486
	N = 400	0.0944	0.0626	0.028	0.1706	0.1166	0.0486	0.18	0.1236	0.0482

Table A2: Rejection frequencies of the z-test, H_0: $\gamma_1 = 2$.

		$\beta_{22} = 1$			$\beta_{22} = 0.1$			$\beta_{22} = 0.0001$		
γ_2	Nominal size	10%	5%	1%	10%	5%	1%	10%	5%	1%
−6	N = 2000	0.0874	0.0542	0.0216	0.2668	0.2282	0.1668	0.5634	0.5166	0.4388
	N = 400	0.1036	0.0702	0.0406	0.4466	0.4058	0.3358	0.57	0.5276	0.451
−3	N = 2000	0.0968	0.0486	0.0128	0.1588	0.1248	0.0802	0.3724	0.3128	0.2164
	N = 400	0.0826	0.052	0.0198	0.2818	0.2362	0.1602	0.3756	0.3162	0.2214
−1.5	N = 2000	0.0922	0.0462	0.0104	0.1058	0.0726	0.033	0.1584	0.102	0.0408
	N = 400	0.0976	0.05	0.011	0.1422	0.0946	0.0416	0.1628	0.1046	0.0408
−0.5	N = 2000	0.0938	0.0464	0.01	0.0676	0.0318	0.0062	0.0124	0.0034	0
	N = 400	0.1004	0.0486	0.0078	0.0412	0.0148	0.0016	0.0142	0.0044	0.0002
1.5	N = 2000	0.0994	0.0514	0.0102	0.0904	0.058	0.023	0.0712	0.0368	0.0078
	N = 400	0.0956	0.0412	0.0096	0.084	0.051	0.0174	0.0736	0.0406	0.0092
3	N = 2000	0.0986	0.051	0.0106	0.1558	0.1214	0.0758	0.3184	0.2576	0.1604
	N = 400	0.083	0.0436	0.0154	0.2488	0.1966	0.1208	0.3266	0.267	0.163
6	N = 2000	0.0922	0.051	0.0188	0.2648	0.2252	0.1654	0.549	0.502	0.421
	N = 400	0.1044	0.0702	0.0324	0.4402	0.3968	0.324	0.5604	0.5144	0.4304