Necessary and sufficient conditions for nonlinear parametric function identification

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Outline

Motivation

- We look into identification conditions of a general **nonlinear function** $\beta(\theta)$ in terms of another **nonlinear function** $\gamma(\theta)$ which is identifiable in **nonlinear models** called **parametric function identification**. The literature on this topic is surprisingly scarce.
  - This paper focuses on point identification of parametric functions which is an extension to the partial identification in the sense of Phillips (1989) and Bekker and Wansbeek (2003) but is different from set-valued identification by Manski (2003) and Tamer (2010).
  - It covers moment conditions as a special case: $\gamma(\theta) = \mathbb{E}[g(Y_t;\theta)], \quad \gamma(\theta) = \mathbb{E}[g(Y_t;\theta)|Z_t]$;
  - It includes semiparametric setups where the distribution of observed variables depends also on nuisance parameters which have an infinite number.
  - It includes known results on identification of the simultaneous equations models (SEMs), the dynamic stochastic general equilibrium (DSGE) models and the likelihood models as special cases.

Motivation (continued)

- The earliest known studies on identification problem is attributed to Lenoir (1913); see Fox (1968), Christ (1985) and Stock and Trebbi (2003). If there is a one-to-one function between the parameter θ and the probability distribution P_θ, then θ is (globally) identifiable.
- The classical papers usually investigate complete identification of θ in the simultaneous equations models; see Fisher (1959, 1961, 1963, 1965, 1966), Rothenberg (1971), Bowden (1973), Richmond (1974), Bekker and Wansbeek (2003), etc. However, the assumptions under which their results hold differ.

Motivation (continued)

- Bekker and Wansbeek (2003) look into identification of a scalar parameter in the likelihood framework; see also Rothenberg (1971). Fisher (1966) considers identification of a single equation in the SEMs and Richmond (1974) focuses on global identification of linear SEMs. But none of them studies identification of a general nonlinear function of θ.
- The research on identification of the New Keynesian Phillips Curves (NKPC) is numerous but no paper focuses on identification of a function of deep parameters [see Dufour, Khalaf and Kichian (2006), Nason and Smith (2008), Kleibergen and Mavroeidis (2008), Ravenna aand Walsh (2008), Calstrom, Fuerst and Paustian (2009), Dees, Pesaran, Smith and Smith (2009), etc.]:

$$\pi_t = \gamma_f \mathbf{E}_t \pi_{t+1} + \gamma_b \pi_{t-1} + \lambda y_t + u_t.$$

Motivation (continued)

- Komunjer and Ng (2011) study identification of the parameter vector θ and a subvector of θ in the DSGE models by An and Schorfheide (2007):

$$X_{t+1} = A(\theta)X_t + B(\theta)u_{t+1}$$
$$Y_{t+1} = C(\theta)X_t + D(\theta)u_{t+1}.$$

- The rank conditions by Iskrev (2010) in the DSGE models look into identification of θ and hold under different assumptions in comparison to Komunjer and Ng (2011); see also Kim (2003), Cochrane (2007), Canova and Sala (2009) for identification issues of the DSGE models. But it does not study identification of a function of θ.
- Existing literature investigates local identification only at given parameter values. We introduce local identification around a point to justify the existence of a consistent estimator.

Motivation (continued)

- Chen, Chernozhukov, Lee and Newey (2011) consider only sufficient condition through the rank of the Jacobian matrix of the moment equations.
- Identification of nonparametric models is not covered in the paper and related literature includes Brown and Matzkin (1998), Chesher (2003), Newey and Powell (2003), Chernozhukov, Imbens and Newey (2007), Imbens and Newey (2009), Florens, Johannes and Van(2012), Matzkin (2012), etc.

Simple illustrations on identification failure

The following linear SEMs are just given for illustration purpose and our paper considers a more general and complicated framework focusing on identification of nonlinear functions of parameters in nonlinear models.

Example I: Simple simultaneous equations models (Maddala (1977))

- Identified supply and demand functions

$$Q_t = \alpha_1 + \beta_1 P_t + \gamma_1 Y_t + u_{t1} \text{ (demand function)},$$
$$Q_t = \alpha_2 + \beta_2 P_t + \gamma_2 R_t + u_{t2} \text{ (supply function)}.$$

The reduced form equations become

$$Q_t = \pi_1 + \pi_2 Y_t + \pi_3 R_t + v_{t1},$$
$$P_t = \pi_4 + \pi_5 Y_t + \pi_6 R_t + v_{t2},$$

where $\pi_1 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_2 - \beta_1}$, $\pi_2 = \frac{\beta_2\gamma_1}{\beta_2 - \beta_1}$, $\pi_3 = -\frac{\beta_1\gamma_2}{\beta_2 - \beta_1}$, $\pi_4 = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1}$, $\pi_5 = \frac{\gamma_1}{\beta_2 - \beta_1}$, $\pi_6 = -\frac{\gamma_2}{\beta_2 - \beta_1}$.

Simple illustrations on identification failure (continued)

- Identified supply function but **unidentified** demand function

$$Q_t = \alpha_1 + \beta_1 P_t + \gamma_1 Y_t + u_{t1} \text{ (demand function)},$$
$$Q_t = \alpha_2 + \beta_2 P_t + u_{t2} \text{ (supply function)}.$$

The reduced form then becomes

$$Q_t = \pi_1 + \pi_2 Y_t + v_{t1},$$
$$P_t = \pi_4 + \pi_5 Y_t + v_{t2}.$$

We cannot identify all 5 structural parameters. Specifically, α_2 and β_2 are identifiable but there is no unique solution to α_1, β_1 and γ_1.

Simple illustrations on identification failure (continued)

- Identified demand function but **unidentified** supply function

$$Q_t = \alpha_1 + \beta_1 P_t + u_{t1} \text{ (demand function)},$$
$$Q_t = \alpha_2 + \beta_2 P_t + \gamma_2 R_t + u_{t2} \text{ (supply function)}.$$

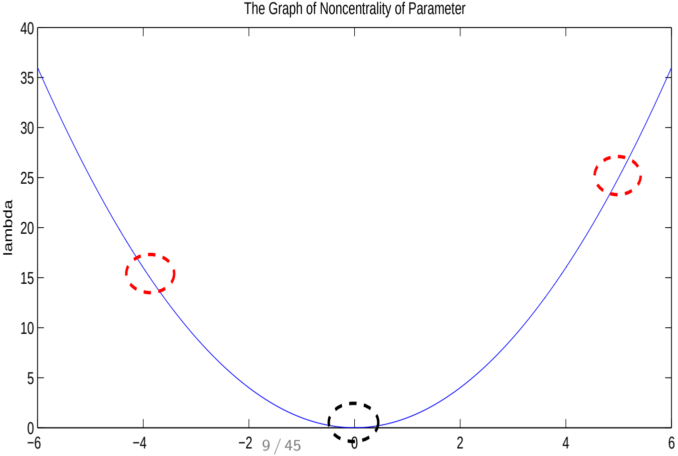
The reduced form is

$$Q_t = \pi_1 + \pi_3 R_t + v_{t1},$$
$$P_t = \pi_4 + \pi_6 R_t + v_{t2}.$$

Now α_1 and β_1 become identifiable but there is no unique solution to α_2, β_2 and γ_2.

Simple illustrations on identification failure (continued)

- **Example II: Local and global identification failure**
  Consider a random variable $X \sim \mathrm{N}(\mu, 1)$. Suppose that the observed data is X^2 which follows $\chi^2(1, \lambda)$ with the noncentrality parameter $\lambda = \mu^2$ and $\mathbb{E}(X^2) = 1 + \lambda$.



Simple illustrations on identification failure (continued)

- **Example III: Identification failure in MA(1) model**
  Consider a MA(1) process:

$$Y_t = \epsilon_t + \theta\epsilon_{t-1},\ \epsilon_t \sim \mathrm{N}[0, \sigma^2].$$

Since

$$\mathrm{Var}(Y_t) = \sigma^2(1+\theta^2),\ \mathrm{Cov}(Y_t Y_{t-1}) = \sigma^2\theta,$$

we have

$$\theta^2 - \frac{\mathrm{Var}(Y_t)}{\mathrm{Cov}(Y_t Y_{t-1})}\theta + 1 = 0 \tag{1}$$

We cannot identify θ from θ^{-1}, since both parameters are solutions for (1). However, we can identify autocorrelation which is a nonlinear function of θ

$$\rho = \frac{\theta}{1+\theta^2}.$$

Questions of interest

The questions of interest are as follows:

- Does there exist a general condition that can include the established results on local (or global) identification as special cases?
- What are the necessary and sufficient conditions for parametric function identification?
- What are the necessary and sufficient conditions for local identification at and around a point?
- How to apply such conditions to nonlinear models, such as the Dynamic Stochastic General Equilibrium models (DSGE)?

Contributions

- Propose a general necessary and sufficient condition for local identification of parametric functions.
- Establish both necessary and sufficient conditions for parametric function identification and provide such conditions subject to restrictions.
- Distinguish between the concept of local identification around θ_0 and the usual definition of local identification at θ_0.
- Give alternative equivalent conditions for parametric function identification using vector space properties.
- Apply our general identification conditions to widely-used statistical and macroeconomic models.

Fundamental concepts

Definition (Global identification of a parametric function at θ_0)

The parametric function $\gamma : \Theta \to \Gamma$ is globally identifiable at $\theta_0 \in \Theta$ if and only if

$$\left(\gamma(\theta) \neq \gamma(\theta_0)\right) \Rightarrow \left(\mathsf{P}_{(\theta,\nu_1)} \neq \mathsf{P}_{(\theta_0,\nu_2)}\right), \ \forall \theta \in \Theta, \ \forall \nu_1, \ \nu_2 \in \Xi.$$

Definition (Global identification of a parametric function over Θ)

The parametric function $\gamma : \Theta \to \Gamma$ is globally identifiable over Θ if and only if

$$\left(\gamma(\theta_1) \neq \gamma(\theta_2)\right) \Rightarrow \left(\mathsf{P}_{(\theta_1,\nu_1)} \neq \mathsf{P}_{(\theta_2,\nu_2)}\right), \ \forall \theta_1, \ \theta_2 \in \Theta, \ \forall \nu_1, \ \nu_2 \in \Xi.$$

Fundamental concepts (continued)

Definition (**Local identification** of a parametric function **at** θ_0)

The parametric function $\gamma : \Theta \to \Gamma$ is locally identifiable at $\theta_0 \in \Theta$ iff there exists an open neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that

$$\left(\gamma(\theta) \neq \gamma(\theta_0)\right) \Rightarrow \left(\mathsf{P}_{(\theta,\nu_1)} \neq \mathsf{P}_{(\theta_0,\nu_2)}\right), \ \forall \theta \in \mathcal{V}(\theta_0), \ \forall \nu_1, \nu_2 \in \Xi.$$

Definition (**Local identification** of a parametric function **around** θ_0)

The parametric function $\gamma : \Theta \to \Gamma$ is locally identifiable around $\theta_0 \in \Theta$ iff there exists an open neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that

$$\left(\gamma(\theta_1) \neq \gamma(\theta_2)\right) \Rightarrow \left(\mathsf{P}_{(\theta_1,\nu_1)} \neq \mathsf{P}_{(\theta_2,\nu_2)}\right), \ \forall \theta_1, \theta_2 \in \mathcal{V}(\theta_0), \ \forall \nu_1, \nu_2 \in \Xi.$$

Fundamental concepts (continued)

Definition (Local identification in terms of another parametric function at and around θ_0)

Let $\beta : \Theta \to B$ and $\gamma : \Theta \to \Gamma$ be two parametric functions. $\beta(\theta)$ is locally identifiable in terms of $\gamma(\theta)$ at $\theta_0 \in \Theta$ iff

$$\left(\beta(\theta) \neq \beta(\theta_0)\right) \Rightarrow \left(\gamma(\theta) \neq \gamma(\theta_0)\right), \quad \forall \theta \in \mathcal{V}(\theta_0).$$

$\beta(\theta)$ is locally identifiable in terms of $\gamma(\theta)$ around $\theta_0 \in \Theta$ iff

$$\left(\beta(\theta_1) \neq \beta(\theta_2)\right) \Rightarrow \left(\gamma(\theta_1) \neq \gamma(\theta_2)\right), \quad \forall \theta_1,\ \theta_2 \in \mathcal{V}(\theta_0).$$

The advantages of introducing the concept of local identification around a point

- Known results on local identification at θ_0 are still restrictive and maybe unsatisfactory for statistical inference. Given the conditions for local identification only at a specific point, we are not sure whether or not an estimator is locally identifiable.
- Global identification over the parameter space is desirable but very difficult to achieve, especially for nonlinear models.
- Local identification around a point lies between the above two identifications. It helps justify that a consistent estimator could be locally identifiable.

Relationships among different concepts of identifications

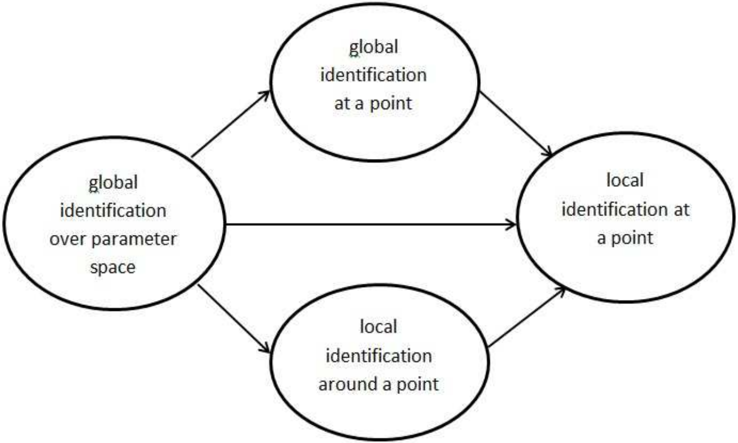
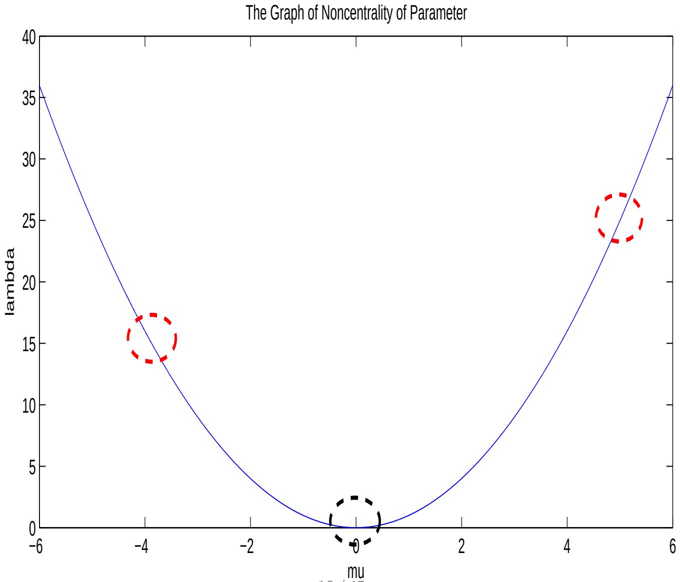


Illustration of different identification concepts



Necessary and sufficient conditions for local identification at θ_0

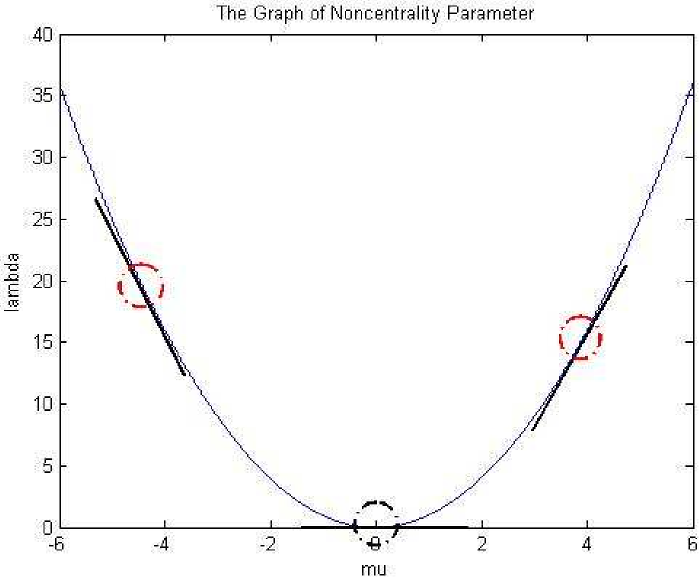
Theorem

Let $\theta_0 \in \Theta$ and $\gamma : \Theta \mapsto \mathbb{R}^G$ be a parametric function which is locally identifiable at θ_0. Suppose $\gamma(\theta)$ is **differentiable at** θ_0 *with Jacobian matrix $J_\gamma(\theta_0)$. Then the condition*

$$\mathrm{rank}[J_\gamma(\theta_0)] = k$$

implies that θ is locally identifiable at θ_0. Furthermore, suppose $\gamma(\theta)$ is **continuously differentiable in some open neighborhood of** θ_0 *with Jacobian matrix $J_\gamma(\theta)$ and θ_0 is a* **regular point** *of $J_\gamma(\theta)$. Then θ is locally identifiable at θ_0 iff $\mathrm{rank}[J_\gamma(\theta_0)] = k$.*

Illustration of the theorem on local identification at a point



Necessary and sufficient conditions for local identification **around** θ_0

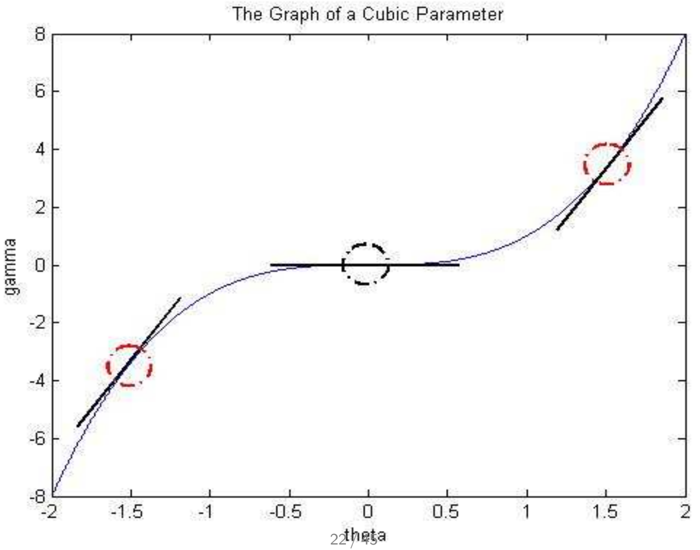
Theorem

Let $\theta_0 \in \Theta$ and $\gamma : \Theta \mapsto \mathbb{R}^G$ be a parametric function which is locally identifiable around θ_0. Suppose $\gamma(\theta)$ is **differentiable in some open neighborhood of θ_0 with continuous differentiability at θ_0**. *Then the condition*

$$\mathrm{rank}[J_\gamma(\theta_0)] = k$$

implies that θ is locally identifiable around θ_0. Furthermore, suppose $\gamma(\theta)$ is **continuously differentiable in some open neighborhood of θ_0** *and θ_0 is a* **regular point** *of $J_\gamma(\theta)$. Then θ is locally identifiable around θ_0 iff $\mathrm{rank}[J_\gamma(\theta_0)] = k$.*

Illustration of the theorem on local identification around a point



Identification of general parametric functions

- Parametric function identification means that the vector θ may not be identifiable but a function of θ can still be identified.
- It is an extension to the identification of a subvector; see Phillips (1989) and Bekker and Wansbeek (2003).
- It differs from set-valued identification by Manski (2003) and Tamer (2010).

Recall a column space of an $m \times n$ matrix A is defined as $\mathrm{Im}(A) \equiv \{y \in \mathbb{R}^m : Ax = y \text{ for some } x \in \mathbb{R}^n\}$.

Identification of general parametric functions (continued)

Theorem (**Sufficient conditions** for local parametric function identification **around** θ_0)

Let $\gamma : \mathbb{R}^k \mapsto \mathbb{R}^G$ *and* $\beta : \mathbb{R}^k \mapsto \mathbb{R}^H$ *be two parametric functions. Assume* $\gamma(\theta)$ *is locally identifiable around* θ_0*. Suppose* $\gamma(\theta)$ *and* $\beta(\theta)$ *are* **differentiable in some neighborhood of** θ_0 **with continuous differentiability at** θ_0*. Then the condition* $\mathrm{Im}\big(J_\beta(\theta_0)'\big) \subseteq \mathrm{Im}\big(J_\gamma(\theta_0)'\big)$ *implies that* $\beta(\theta)$ *is locally identifiable in terms of* $\gamma(\theta)$ *around* θ_0.

Identification of general parametric functions (continued)

Theorem (**Necessary and sufficient conditions** for local parametric function identification **around** θ_0)

Let $\gamma : \mathbb{R}^k \mapsto \mathbb{R}^G$ *and* $\beta : \mathbb{R}^k \mapsto \mathbb{R}^H$ *be two parametric functions. Assume* $\gamma(\theta)$ *is locally identifiable around* θ_0*. Suppose* $\gamma(\theta)$ *and* $\beta(\theta)$ *are* **continuously differentiable in an open neighborhood of** θ_0 *and* θ_0 *is a* **regular point** *of both* $J_\gamma(\theta)$ *and* $J_\beta(\theta)$*. Then* $\beta(\theta)$ *is locally identifiable in terms of* $\gamma(\theta)$ *around* θ_0 *if and only if* $\mathrm{Im}\big(J_\beta(\theta_0)'\big) \subseteq \mathrm{Im}\big(J_\gamma(\theta_0)'\big)$.

A simple case on parametric function identification

A SEMs setup with one IV and one endogenous regressor

$$y = \beta Y + u, \tag{2}$$

$$Y = \pi Z + v, \tag{3}$$

where both β and π are scalars. Plugging (3) into (2) gives

$$y = \lambda Z + (\beta v + u),$$

where $\lambda = \beta\pi$. Hence β is identifiable if and only if $\pi \neq 0$. Let

$$\theta = \begin{bmatrix} \beta \\ \pi \end{bmatrix}, \ \gamma(\theta) = \begin{bmatrix} \pi \\ \lambda \end{bmatrix}, \ \beta(\theta) = \beta.$$

A simple case on parametric function identification (continued)

$$J_\gamma(\theta) = \begin{bmatrix} 0 & 1 \\ \pi & \beta \end{bmatrix}, \; J_\beta(\theta) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Hence

$$\begin{aligned} &\left(\mathrm{Im} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \subseteq \mathrm{Im} \begin{bmatrix} 0 & \pi \\ 1 & \beta \end{bmatrix} \right) \\ \iff & \; (\pi \neq 0) \, . \end{aligned} \tag{4}$$

$\begin{bmatrix} \beta & \pi \end{bmatrix}'$ is a regular point of both $J_\gamma(\theta)$ and $J_\beta(\theta)$, $\forall \pi \neq 0$. (4) is both necessary and sufficient for identification of β.

Alternative formulae for function identification

Suppose $\gamma(\theta)$ and $\beta(\theta)$ are continuously differentiable in $\mathcal{V}(\theta_0)$ and θ_0 is a regular point of $J_\gamma(\theta)$ and $J_\beta(\theta)$. Given $\gamma(\theta)$ is locally identifiable around θ_0, we give a group of equivalent conditions for local identification of $\beta(\theta)$ around θ_0, some of which are listed for demonstration purpose.

$$\mathrm{Im}\big(J_\beta(\theta_0)'\big) \subseteq \mathrm{Im}\big(J_\gamma(\theta_0)'\big);$$

$$J_\beta(\theta_0) = F(\theta_0)J_\gamma(\theta_0), \text{ for some matrix } F(\theta_0); \tag{5}$$

$$\mathrm{rank}\begin{bmatrix} J_\gamma(\theta_0) \\ J_\beta(\theta_0) \end{bmatrix} = \mathrm{rank}(J_\gamma(\theta_0)). \tag{6}$$

Alternative formulae for function identification (continued)

$$\ker[J_\gamma(\theta_0)] \subseteq \ker[J_\beta(\theta_0)]\,;$$

$$\operatorname{rank}\begin{bmatrix} J_\gamma(\theta_0) \\ J_\beta(\theta_0) + V_1(\theta_0)J_\gamma(\theta_0) \end{bmatrix} = \operatorname{rank}(J_\gamma(\theta_0)),$$

for any $q \times p$ matrix $V_1(\theta_0)$;

$$\operatorname{rank}\begin{bmatrix} J_\gamma(\theta_0) + V_2(\theta_0)J_\beta(\theta_0) \\ J_\beta(\theta_0) \end{bmatrix} = \operatorname{rank}(J_\gamma(\theta_0)),$$

for any $p \times q$ matrix $V_2(\theta_0)$;

$$J_\beta(\theta_0) = J_\beta(\theta_0)J_\gamma(\theta_0)^- J_\gamma(\theta_0),$$

for some g-inverse $J_\gamma(\theta_0)^-$.

Two examples on alternative formulae

- One IV simple SEMs setup with $\pi \neq 0$
  Then

$$J_\beta(\theta) = \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{\beta}{\pi} & \frac{1}{\pi} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \pi & \beta \end{bmatrix} = \begin{bmatrix} -\frac{\beta}{\pi} & \frac{1}{\pi} \end{bmatrix} J_\gamma(\theta).$$

  So (5) holds. Also

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} J_\gamma(\theta) \\ J_\beta(\theta) \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} 0 & 1 \\ \pi & \beta \\ 1 & 0 \end{bmatrix} = 2 \\ &= \operatorname{rank}\begin{bmatrix} 0 & 1 \\ \pi & \beta \end{bmatrix} \\ &= \operatorname{rank}(J_\gamma(\theta)). \end{aligned}$$

  (6) holds.

Two examples (continued)

- Identified supply function but unidentified demand function

$$
\begin{aligned}
Q_t &= \alpha_1 + \beta_1 P_t + \gamma_1 Y_t + u_{t1} \text{ (demand function)},\\
Q_t &= \alpha_2 + \beta_2 P_t + u_{t2} \text{ (supply function)}.
\end{aligned}
$$

The reduced form equations

$$
\begin{aligned}
Q_t &= \pi_1 + \pi_2 Y_t + v_{t1},\\
P_t &= \pi_3 + \pi_4 Y_t + v_{t2},
\end{aligned}
$$

$\pi_1 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\beta_2 - \beta_1}, \quad \pi_2 = \frac{\gamma_1\beta_2}{\beta_2 - \beta_1}, \quad \pi_3 = \frac{\alpha_1 - \alpha_2}{\beta_2 - \beta_1}, \quad \pi_4 = \frac{\gamma_1}{\beta_2 - \beta_1}.$
Hence $\theta = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \alpha_2 & \beta_2 \end{bmatrix}'$,
$\gamma(\theta) = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}'$, $\beta_d(\theta) = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \end{bmatrix}'$,
$\beta_s(\theta) = \begin{bmatrix} \alpha_2 & \beta_2 \end{bmatrix}'$.

Two examples (continued)

θ is a regular point of $J_\gamma(\theta)$ if $\beta_1 \neq \beta_2$, $\beta_1 \neq 0$ and $\gamma_1 \neq 0$.

$$J_\gamma(\theta) = \frac{1}{\beta_2 - \beta_1} \begin{bmatrix} \beta_2 & -\frac{(\alpha_2-\alpha_1)\beta 2}{\beta_2-\beta_1} & 0 & -\beta_1 & \frac{(\alpha_2-\alpha_1)\beta 1}{\beta_2-\beta_1} \\ 0 & \frac{\beta_2\gamma_1}{\beta_2-\beta_1} & \beta_2 & 0 & -\frac{\beta_1\gamma_1}{\beta_2-\beta_1} \\ 1 & -\frac{\alpha_2-\alpha_1}{\beta_2-\beta_1} & 0 & -1 & \frac{\alpha_2-\alpha_1}{\beta_2-\beta_1} \\ 0 & \frac{\gamma_1}{\beta_2-\beta_1} & 1 & 0 & -\frac{\gamma_1}{\beta_2-\beta_1} \end{bmatrix},$$

$$J_{\beta_d}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \; J_{\beta_s}(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\mathrm{rank}\left(J_\gamma(\theta)\right) = 4 = \mathrm{rank}\begin{bmatrix} J_\gamma(\theta) \\ J_{\beta_s}(\theta) \end{bmatrix}, \; \mathrm{rank}\begin{bmatrix} J_\gamma(\theta) \\ J_{\beta_d}(\theta) \end{bmatrix} = 5.$$

- The demand function is unidentified.
- The supply function is identified.
- We can identify α_1 and γ_1 but not β_1 in demand function.

Discussions on alternative formulae

- All the equivalent conditions hold for local identification at θ_0.
- The connections among these alternative conditions and the meaning of each condition are discussed in detail in Dufour and Liang (2012).
- The elements of $J_\gamma(\theta)$ and $J_\beta(\theta)$ are generally nonlinear transformations of θ.
- In some cases, these elements can be independent of θ or linear functions of θ. Then the equivalent conditions become global.

Parametric function identification with restrictions

Suppose the parameters are restricted as $\xi(\theta_0) = c$, where $\xi : \mathbb{R}^k \mapsto \mathbb{R}^R$.

Corollary (Conditions for local identification with restrictions)

Suppose $\gamma(\theta)$, $\beta(\theta)$ and $\xi(\theta)$ are continuously differentiable in an open neighborhood of θ_0 and θ_0 is a regular point of $J_\gamma(\theta)$, $J_\beta(\theta)$ and $J_\xi(\theta)$. Suppose $\gamma(\theta)$ and $\xi(\theta)$ are locally identifiable around θ_0. Then

$$\mathrm{Im}\big(J_\beta(\theta_0)'\big) \subseteq \mathrm{Im}\left[\begin{array}{cc} J_\gamma(\theta_0)' & J_\xi(\theta_0)' \end{array}\right]$$

is both necessary and sufficient for $\beta(\theta)$ to be locally identifiable around θ_0.

An illustration on parametric function identification with restrictions

Consider again

$$Q_t = \alpha_1 + \beta_1 P_t + \gamma_1 Y_t + u_{t1} \text{ (demand function)},$$
$$Q_t = \alpha_2 + \beta_2 P_t + u_{t2} \text{ (supply function)}.$$

Impose $\xi(\theta) = \gamma_1 + \beta_2 = c$. Then $J_\xi(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ and

$$\operatorname{rank} \begin{bmatrix} J_\gamma(\theta) \\ J_\xi(\theta) \\ J_{\beta_d}(\theta) \end{bmatrix} = 5 = \operatorname{rank} \begin{bmatrix} J_\gamma(\theta) \\ J_\xi(\theta) \end{bmatrix}.$$

The demand function becomes identifiable.

The simultaneous equations models (SEMs)

We provide the general rank condition for local identification of the (restricted) structural parameters θ around a given point. In contrast, the known results in the literature are concerned with local identification at a point and may not be very useful for statistical inference.

Consider a system of G simultaneous equations:

$$YB + X\Gamma = U,$$

where Y is endogenous, X is exogenous and B and Γ are matrices of unknown coefficients. Denote the vector of all the structural parameters

$$\theta \equiv \begin{pmatrix} (\mathrm{vec}(B))' & (\mathrm{vec}(\Gamma))' & (\mu)' & (\mathrm{vec}(\Sigma))' \end{pmatrix}'.$$

SEMs (continued)

Suppose there exist M continuously differentiable constraints on θ

$$\phi(\theta) = 0$$

whose Jacobian matrix is denoted $J_\phi(\theta)$. Define the moment functions

$$m_1(\theta) \equiv \begin{bmatrix} B' & \Gamma' \end{bmatrix} \begin{bmatrix} \mu_{Y_t} \\ \mu_{X_t} \end{bmatrix} - \mu = 0$$

and

$$m_2(\theta) \equiv \begin{bmatrix} B' & \Gamma' \end{bmatrix} \begin{bmatrix} \Sigma_{Y_t} & \Sigma_{Y_t X_t} \\ \Sigma_{X_t Y_t} & \Sigma_{X_t} \end{bmatrix} \begin{bmatrix} B \\ \Gamma \end{bmatrix} - \Sigma = 0.$$

We obtain the necessary and sufficient condition for identification of an arbitrary parametric function $\beta(\theta)$ according to the general rank condition (6) is as follows:

$$\mathrm{rank} \begin{bmatrix} J_m(\theta_0) \\ J_\phi(\theta_0) \\ J_\beta(\theta_0) \end{bmatrix} = \mathrm{rank} \begin{bmatrix} J_m(\theta_0) \\ J_\phi(\theta_0) \end{bmatrix}. \tag{7}$$

Generalization of Rothenberg (1971)'s SEMs

Rothenberg assumes $\mathbb{E}(U_t) = 0$ for simplicity and X has full column rank and B is nonsingular. Denote the $G^2 + GK$ reduced form parameters as

$$\psi(\theta) = \begin{pmatrix} \mathrm{vec}(\Pi(\theta)) \\ \mathrm{vec}(\Omega(\theta)) \end{pmatrix}.$$

and the Jacobian matrix as $J_\psi(\theta)$. Set

$$\gamma(\theta) = \psi(\theta), \quad \xi(\theta) = \phi(\theta), \quad \beta(\theta) = \theta.$$

Applying our rank condition (6) or (7), we achieve the necessary and sufficient condition for the local identification of θ at θ_0

$$\mathrm{rank} \begin{bmatrix} J_\psi(\theta_0) \\ J_\phi(\theta_0) \\ I_{2G^2+GK} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} J_\psi(\theta_0) \\ J_\phi(\theta_0) \end{bmatrix}$$

which implies the result of Rothenberg (1971, Theorem 9):

$$\mathrm{rank} \begin{bmatrix} J_\psi(\theta_0) \\ J_\phi(\theta_0) \end{bmatrix} = 2G^2 + GK.$$

Generalization of Fisher (1966)'s SEMs

Identification for a scalar parameter in SEMs of Fisher can be checked by using our established conclusions. Without loss of generality, we can identify the first structural parameter θ_1 and set

$$\beta(\theta) = \theta_1.$$

The necessary and sufficient condition for local identification of θ_1 is

$$\operatorname{rank}\begin{bmatrix} J_\psi(\theta_0) \\ J_\phi(\theta_0) \\ e_1' \end{bmatrix} = \operatorname{rank}\begin{bmatrix} J_\psi(\theta_0) \\ J_\phi(\theta_0) \end{bmatrix}.$$

Fisher (1966, Theorem 6.4.1) proposes a similar condition under the stronger assumption that θ_0 is a normal point rather than a regular point.

Generalization of Bekker and Wansbeek (2003)'s SEMs

Suppose

$$A\mathcal{K}(\theta_0) = 0,$$

where

$$A \equiv \begin{bmatrix} J_\psi(\theta_0) \\ J_\phi(\theta_0) \end{bmatrix}$$

and $\mathcal{K}(\theta_0)$ is a basis of $\ker(A)$. If $e_1'\mathcal{K}(\theta_0) = 0$, then $\mathcal{K}(\theta_0)$ is also the basis of $\ker \begin{bmatrix} A \\ e_1' \end{bmatrix}$. The result of Bekker and Wansbeek (2003, Corollary 1) reduces to

$$\ker \begin{bmatrix} A \\ e_1' \end{bmatrix} = \ker(A),$$

which is a special case of our general rank condition.

The Dynamic Stochastic General Equilibrium (DSGE) models

Consider the state-space form of the DSGE models:

$$X_{t+1} = A(\theta)X_t + B(\theta)u_{t+1}$$

and

$$Y_{t+1} = C(\theta)X_t + D(\theta)u_{t+1},$$

where θ is a $k \times 1$ vector of deep or structural parameters, u is the structural shocks, A is $n \times n$, B is $n \times m$, C is $p \times n$ and D is $p \times m$.

Applying our general rank condition to the DSGE models, we can show that it generalizes the major local identification conditions by Iskrev (2010) and Komunjer and Ng (2011).

The DSGE models (continued)

Denote $\Lambda(\theta)$ as the Jacobian of the vector

$$\lambda(\theta,\ S,\ T) = \left(\begin{array}{c} \mathrm{vec}\big(SA(\theta)S^{-1}\big) \\ \mathrm{vec}(SB(\theta)T) \\ \mathrm{vec}\big(C(\theta)S^{-1}\big) \\ \mathrm{vec}(D(\theta)T) \\ \mathrm{vech}\Big(T^{-1}\Sigma(\theta)T^{-1\prime}\Big) \end{array} \right).$$

S and T are non-singular matrices that link two sets of coefficients $ABCD$ in a similar transformation so that $g(a;\theta_1) = g(a;\theta_2)T$. For notation purpose, denote

$$\widetilde{W} = \begin{bmatrix} S & O \\ O & T \end{bmatrix}, \quad \mathrm{vec}W = \begin{bmatrix} \mathrm{vec}S \\ \mathrm{vec}T \end{bmatrix},$$

where $\mathrm{vec}W$ is a $j \times 1$ vector.

The DSGE models (continued)

Let's set

$$\beta(\theta) = \begin{bmatrix} \theta \\ \mathrm{vec}\, W \end{bmatrix},$$

it follows from the general rank condition (6) that the necessary and sufficient condition for local identification of θ is

$$\mathrm{rank} \begin{bmatrix} J_{\lambda,\theta}(\theta_0,\ I_j) & J_{\lambda,W}(\theta_0,\ I_j) \\ I_k & O \\ O & I_j \end{bmatrix} = \mathrm{rank} \begin{bmatrix} J_{\lambda,\theta}(\theta_0,\ I_j) & J_{\lambda,W}(\theta_0,\ I_j) \end{bmatrix}$$

which entails that

$$\mathrm{rank} \begin{bmatrix} J_{\lambda,\theta}(\theta_0,\ I_j) & J_{\lambda,W}(\theta_0,\ I_j) \end{bmatrix} = k + j. \tag{8}$$

The DSGE models (continued)

Furthermore, denote θ_q as a q dimension subvector of θ and θ_{k-q} the rest of the elements of θ and the partial derivatives $J_{\lambda,\theta_q}(\theta)$ and $J_{\lambda,\theta_{k-q}}(\theta)$ respectively. If we set

$$\beta(\theta) = \begin{bmatrix} \theta_q \\ \mathrm{vec}\, W \end{bmatrix},$$

the necessary and sufficient condition for identification of θ_q becomes

$$\begin{aligned} &\operatorname{rank} \begin{bmatrix} J_{\lambda,\theta_q}(\theta_0,\ I_j) & J_{\lambda,\theta_{k-q}}(\theta_0,\ I_j) & J_{\lambda,W}(\theta_0,\ I_j) \\ I_q & O & O \\ O & O & I_j \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} J_{\lambda,\theta_q}(\theta_0,\ I_j) & J_{\lambda,\theta_{k-q}}(\theta_0,\ I_j) & J_{\lambda,W}(\theta_0,\ I_j) \end{bmatrix} \\ &= (q+j) + \operatorname{rank}\left(J_{\lambda,\theta_{k-q}}(\theta_0,\ I_j)\right). \end{aligned} \tag{9}$$

Conclusion

- We establish both necessary and sufficient conditions for nonlinear parametric function identification of nonlinear models subject to restrictions.
- We define local identification around θ_0, which is important for statistical inference.
- We demonstrate the generality of the proposed necessary and sufficient conditions by applying them to statistical and macroeconomic models, namely the SEMs, the DSGE models and the likelihood models.