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EXACT TESTS AND CONFIDENCE SETS IN LINEAR REGRESSIONS WITH AUTOCORRELATED ERRORS

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This paper proposes a general method to build exact tests and confidence sets in linear regressions with first-order autoregressive Gaussian disturbances. Because of a nuisance parameter problem, we argue that generalized bounds tests and conservative confidence sets provide natural inference procedures in such a context. Given an exact confidence set for the autocorrelation coefficient, we describe how to obtain a similar simultaneous confidence set for the autocorrelation coefficient and any subvector of regression coefficients. Conservative confidence sets for the regression coefficients are then deduced by a projection method. For any hypothesis that specifies jointly the value of the autocorrelation coefficient and any set of linear restrictions on the regression coefficients, we get exact similar tests. For testing linear hypotheses about the regression coefficients only, we suggest bounds-type procedures. Exact confidence sets for the autocorrelation coefficient are built by "inverting" autocorrelation tests. The method is illustrated with two examples.

KEYWORDS: Autocorrelation, bounds test, conservative confidence set, exact test, first-order autoregressive process, linear regression, nuisance parameter, projection method, union-intersection method.

1. INTRODUCTION

ONE OF THE MOST WIDELY used models in econometrics is the linear regression model with first-order autoregressive errors. This model can be stated as follows:

$$y_t = x_t'\beta + u_t \qquad (t = 1, \ldots, T), \tag{1.1}$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{ind}}{\sim} N(0, \sigma^2),$$

where y_t is the dependent variable (at time t), x_t is a $k \times 1$ vector of fixed regressors, β is a $k \times 1$ vector of fixed coefficients, and u_t is a random disturbance; the coefficients β, ρ, and σ^2 are unknown. Further, one of the two following assumptions is usually made:

ASSUMPTION A: $|\rho| < 1$ *and* $u_1 \sim N[0, \sigma^2/(1-\rho^2)]$.

ASSUMPTION B: $|\rho| \leqslant 1$ *and* u_1 *is fixed* (*or independent of* $\varepsilon_2, \ldots, \varepsilon_T$ *with an arbitrary distribution*).

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The first assumption implies that u_t follows a stationary process (Koopmans (1942)). The second one allows nonstationary processes by letting u_1 follow an arbitrary distribution and/or $|\rho| = 1$.

Several authors have studied inference methods for this model; for a review, see Judge et al. (1985, Chap. 8). In particular, considerable effort has been devoted to comparing the efficiency of alternative estimators. Despite this important work, tests and confidence sets remain based on asymptotic theory. Moreover, various simulation results suggest that asymptotic critical values can be very unreliable, especially when ρ is close to 1; see Park and Mitchell (1980) and Miyazaki and Griffiths (1984). Improvements may be obtained by using asymptotic expansion methods but the latter still do not yield exact inference procedures (Rothenberg (1984a, 1984b), Tse (1980)). Asymptotic expansions may also be very inaccurate, especially again when ρ is close to 1; see Rothenberg (1984a). Indeed, an important insight provided by the finite-sample theory of standard estimators and test statistics is the pervasiveness of the nuisance parameter problem: the distributions of various proposed statistics depend on unknown parameters that are irrelevant for a given inference problem (see Taylor (1983)). For example, the null distribution of a test statistic for a given restriction on β may depend on the unknown value of ρ. Consequently, the distribution of the test statistic for the hypothesis of interest is not uniquely determined, even under the null hypothesis. This suggests that efficient similar tests can be difficult to obtain in such a context.[2]

One early approach to dealing with a problem in which the distribution of the test statistic is difficult to obtain is the well-known Durbin-Watson (DW) bounds test against the autocorrelation of errors in regression models (Durbin and Watson (1950, 1951)). In this case, however, the distribution of the test statistic does not depend on any unknown parameter (under the null hypothesis). Nowadays, it is easy to compute marginal significance levels for DW tests (e.g., with the algorithms proposed by Imhof (1961) or Pan Jie Jian (1968)). The approach taken by DW to test $\rho = 0$ was extended to test hypotheses about the regression coefficients; see Watson (1955), Watson and Hannan (1956), Vinod (1976), Kiviet (1979, 1980), Vinod and Ullah (1981, Ch. 4), Zinde-Walsh and Ullah (1987), Hillier and King (1987). One then considers a standard t or F statistic and finds upper and lower bounds on its distribution (over the space of regressor matrices) for a given covariance matrix of the errors. However, these bounds can be far away from each other and, in important cases, the difference between them can go to infinity: for an AR(1) process, one can see easily from the tables supplied by Vinod (1976) and Kiviet (1980) that the upper bound becomes exceedingly large as ρ increases to one. Indeed, from Watson and Hannan (1956, p. 439) and Vinod (1976, p. 930), it is easy to see that the upper bound in this case tends to infinity as ρ approaches one. Thus, unless one is ready to exclude the important

[2] For example, the methods studied by Hannan (1955) and Krishnaiah and Murthy (1966) lead to cutting by one half (approximately) the effective sample size and do not seem to be applicable when $\rho = 1$ or $|\rho| > 1$. For further discussion of nuisance parameters and similar (versus nonsimilar) tests in time-series models, see Nankervis and Savin (1985).

case where ρ is equal or close to one, the method breaks down. Despite this difficulty, we will draw an important idea from this work: bounds tests may be a way to deal with nuisance parameter problems, even though the particular bounds procedures previously suggested have important practical shortcomings.

In this paper, we propose a general method to obtain exact tests and confidence sets in linear regressions with AR(1) errors. For any hypothesis that specifies jointly the value of the autocorrelation coefficient ρ and any set of linear restrictions on the regression coefficient β, we get exact similar tests. For linear hypotheses about β only, we propose bounds-type procedures. The latter however do not have the problems of previous bounds methods. In particular, the bounds do not explode if one allows $\rho = 1$.

The approach adopted is a three-stage confidence procedure: we first get an exact confidence set for the nuisance parameter ρ, we use it to build a simultaneous confidence set for ρ and the regression coefficients of interest, and finally we apply a union-intersection method to obtain confidence sets for the regression coefficients. These confidence sets are valid irrespective of the true value of ρ. The confidence set (or interval) for ρ is derived from an autocorrelation test by exploiting the duality between tests and confidence sets. The simultaneous confidence set for ρ and the regression coefficients is obtained by combining the confidence set for ρ with the corresponding family of "conditional" optimal confidence sets for the regression coefficients of interest. The tests are derived from this simultaneous set. Given the latter, it is straightforward to obtain an exact test for any null hypothesis that specifies the value of ρ jointly with a number of linear restrictions on β. For hypotheses about β only, the tests take the form of "generalized bounds tests." In contrast with traditional bounds tests based on a single test statistic, we use two test statistics with nested critical regions: the smaller critical region yields a conservative test, the larger one gives a liberal test, while the difference between the two regions may be viewed as an "inconclusive" region. Critical values are based on a central F distribution. There is no special difficulty in dealing with values of ρ equal or close to one. Besides, as a special case of the approach proposed, we describe a method which does not require computing a confidence interval for ρ but simply exploits the condition $|\rho| \leqslant 1$.

In Section 2, we define the notion of a bounds test. In Section 3, we describe how, given a confidence set for ρ, one can construct exact confidence sets and exact tests for the regression coefficients of the model. In Section 4, we discuss the construction of exact confidence sets for ρ. In Section 5, we illustrate the methods proposed. Finally, in Section 6, we make a few concluding remarks.

2. CONSERVATIVE, LIBERAL, AND GENERALIZED BOUNDS TESTS

For the purposes of our discussion, we will find it useful to recall or introduce some terminology. Let Y be a vector of n observations with probability distribution $F(y; \theta)$, $y \in S, \theta \in \Omega$, where S is a subset of the Euclidean space $\mathbb{R}^n$ and θ is a vector of unknown parameters that belong to an admissible set Ω. Let ω be a

subset of Ω. We wish to test the null hypothesis H_0: $\theta \in \omega$. We are especially interested by situations where ω contains more than one element (composite null hypothesis).

Typically, a (nonrandomized) test of H_0 is obtained by defining a partition (R, A) of the sample space S. If $Y \in R$, H_0 is rejected; if $Y \in A$, H_0 is accepted (or not rejected). The size of the test is $\sup_{\theta \in \omega} P_\theta(Y \in R) \equiv \alpha$, where $P_\theta(\cdot)$ is the probability measure corresponding to θ. Note that we can have $P_\theta(Y \in R) < \alpha$ for some $\theta \in \omega$. A test whose *size does not exceed* α has *level* α; see Lehmann (1986, Chapter 3) and Rao (1973, p. 446).

In many problems, it is difficult to find a reasonable test whose size is known. In such cases, one usually tries to find a test such that $P_\theta(Y \in R) \simeq \alpha$ for all $\theta \in \omega$, for example by using an asymptotic approximation. However, in most cases, it is not known whether the true probability of rejection under H_0 (type-I error) is greater or smaller than the stated level. To deal with situations where it is difficult to find tests with known size, we will find it convenient to distinguish the *nominal level* of a test (i.e., the stated level) from its true level and we will define two types of tests whose critical regions have unknown size.

DEFINITION 2.1: Let (R, A) be a test with nominal level α for the null hypothesis H_0: $\theta \in \omega$. Then (R, A) is said to be conservative iff $P_\theta(Y \in R) \leqslant \alpha$ for all $\theta \in \omega$, and liberal iff $P_\theta(Y \in R) \geqslant \alpha$ for all $\theta \in \omega$.

Correspondingly, we can speak of conservative and liberal confidence sets.

DEFINITION 2.2: Let C be a confidence set for θ with nominal level $1 - \alpha$. The confidence set C is conservative iff $P_\theta[\theta \in C] \geqslant 1 - \alpha$, for all $\theta \in \Omega$, and liberal iff $P_\theta[\theta \in C] \leqslant 1 - \alpha$, for all $\theta \in \Omega$.

When H_0 is rejected by a level-α conservative test, we know that the conclusion of the test would remain the same if the critical region R were enlarged to make its size equal to α. Similarly, when H_0 is accepted by a level-α liberal test, the conclusion would remain the same if the critical region R were reduced to make its size equal to α. In the other cases, it is better to consider that the test is inconclusive.

Suppose now that, for a given hypothesis H_0, we can obtain both a conservative test (R_1, A_1) and a liberal test (R_2, A_2), each with nominal level α and such that $R_1 \subseteq R_2$. The condition $R_1 \subseteq R_2$ makes sure that the two tests do not yield conflicting answers. When $Y \in R_2 \setminus R_1$, both tests are inconclusive. This suggests the following procedure: reject H_0 if $Y \in R_1$, accept H_0 if $Y \in A_2$, and consider the test inconclusive otherwise. We call a procedure of this type a *generalized bounds test* with level α.

DEFINITION 2.3: Let R, A, and W be three disjoint subsets of the sample space S such that $R \cup A \cup W = S$. We say that the triplet (R, A, W) describes a

generalized bounds test with level α for the null hypothesis H_0: $\theta \in \omega$ if

$$P_\theta(Y \in R) \leqslant \alpha, \qquad P_\theta(Y \in A) \leqslant 1 - \alpha, \qquad \text{for all } \theta \in \omega;$$

we reject H_0 when $Y \in R$, we accept H_0 when $Y \in A$ and we consider the test inconclusive when $Y \in W$.

A well-known example of a bounds test is the Durbin-Watson (1950, 1951) test against positive autocorrelation in regression. If $d(Y)$ is the DW statistic and $[d_L(\alpha), d_U(\alpha)]$ are the significance bounds, where $d_L(\alpha) < d_U(\alpha)$, the three decision sets of the test are $R = \{Y\colon\ d(Y) \leqslant d_L(\alpha)\}$, $A = \{Y\colon\ d(Y) \geqslant d_U(\alpha)\}$, and $W = \{Y\colon\ d_L(\alpha) < d(Y) < d_U(\alpha)\}$. The bounds test is obtained here by considering the extrema of the distribution function of $d(Y)$ over the space of possible regressor matrices of a given dimension. Other examples of this approach include various extensions of the DW bounds test (e.g., Wallis (1972), Vinod (1973), King (1981, 1985)) and the procedures described by Watson (1955), Watson and Hannan (1956), Vinod (1976), Kiviet (1979, 1980), Zinde-Walsh and Ullah (1987), and Hillier and King (1987).

This is not, however, the only way to obtain a *generalized bounds test* (as defined above). In particular, the bounds tests introduced below are based on two tests statistics, say $D_1(Y)$ and $D_2(Y)$, instead of one: the decision regions have the form $R = \{Y\colon\ D_1(Y) > F_1(\alpha)\}$, $A = \{Y\colon\ D_2(Y) \leqslant F_2(\alpha)\}$, where $D_1(Y) \leqslant D_2(Y)$ with probability 1 and $F_1(\alpha) \geqslant F_2(\alpha)$.

3. INFERENCE FOR REGRESSION COEFFICIENTS

Consider the regression model with AR(1) errors as described by (1.1) and either Assumption A or Assumption B (a more general assumption allowing explosive processes is considered in the Appendix). In this section we study how, given an exact confidence set for the autocorrelation coefficient ρ, we can make exact inferences for the regression coefficient β. The problem of finding an exact confidence set for ρ will be discussed in Section 4.

Suppose first that the coefficient ρ is known. Then, it is easy to transform the model so that the disturbances are uncorrelated. The transformed model has the form

$$y_t(\rho) = x_t(\rho)'\beta + u_t(\rho) \qquad (t = p, \ldots, T), \tag{3.1}$$

where

$$y_t(\rho) = y_t - \rho y_{t-1}, \qquad x_t(\rho) = x_t - \rho x_{t-1}, \qquad u_t(\rho) = u_t - \rho u_{t-1} \quad (t = 2, \ldots, T). \tag{3.2}$$

Under Assumption A, $p = 1$ and

$$y_1(\rho) = \sqrt{1-\rho^2}\, y_1, \qquad x_1(\rho) = \sqrt{1-\rho^2}\, x_1, \qquad u_1(\rho) = \sqrt{1-\rho^2}\, u_1; \tag{3.3}$$

under Assumption B, $p = 2$. Further, when the model contains an intercept, we

have $x_t = (1, z_t')'$ and

$$(3.4) \qquad y_t - \rho y_{t-1} = (1-\rho)\beta_1 + (z_t - \rho z_{t-1})'\bar{\gamma} + \varepsilon_t \qquad (t = 2, \ldots, T),$$

where $\beta = (\beta_1, \gamma')'$, $\bar{\gamma} = (\beta_2, \ldots, \beta_k)'$. We see immediately that the intercept β_1 is not identified when $\rho = 1$ (under Assumption B). To avoid this problem, we will reparameterize the model by defining $\tilde{\beta}_1 = (1-\rho)\beta_1$ under Assumption B. $\tilde{\beta}_1$ is identified for all ρ. We then consider the transformed model

$$(3.5) \qquad y_t - \rho y_{t-1} = \tilde{\beta}_1 + (z_t - \rho z_{t-1})'\bar{\gamma} + \varepsilon_t \qquad (t = 2, \ldots, T).^3$$

Using matrix notation we can write the transformed model

$$(3.6) \qquad y(\rho) = X(\rho)\tilde{\beta} + u(\rho),$$

where

$$(3.7) \qquad y(\rho) = [y_p(\rho), \ldots, y_T(\rho)]', \qquad u(\rho) = [u_p(\rho), \ldots, u_T(\rho)]',$$
$$X(\rho) = [\tilde{x}_p(\rho), \ldots, \tilde{x}_T(\rho)]', \qquad \tilde{\beta} = (\tilde{\beta}_1, \beta_2, \ldots, \beta_k)',$$

and

$$(3.8) \qquad \tilde{x}_t(\rho) = [1, (z_t - \rho z_{t-1})']' \quad \text{and} \quad \tilde{\beta}_1 = (1-\rho)\beta_1,$$
if the model contains an intercept and Assumption B holds,
$$\tilde{x}_t(\rho) = x_t(\rho) \quad \text{and} \quad \tilde{\beta}_1 = \beta_1, \quad \text{otherwise.}$$

In the sequel, we shall also assume that

$$(3.9) \qquad \text{rank}\,[X(\rho)] = k < T_1 = T - p + 1$$

for all admissible values of ρ. T_1 is the effective number of observations. Furthermore, let

$$(3.10) \qquad \hat{\beta}(\rho) = [X(\rho)'X(\rho)]^{-1}X(\rho)'y(\rho), \qquad \hat{u}(\rho) = y(\rho) - X(\rho)\hat{\beta}(\rho),$$
$$s(\rho)^2 = \hat{u}(\rho)'\hat{u}(\rho)/(T_1 - k) = \|\hat{u}(\rho)\|^2/(T_1 - k).$$

Let $\gamma = C\tilde{\beta}$ be an $m \times 1$ vector of linear transformations of $\tilde{\beta}$, where $\text{rank}(C) = m \leqslant k$. We will consider in turn the following problems: (P1) construct a joint confidence set for (ρ, γ); (P2) test H_0: $\rho = \rho_0$, $\gamma = \gamma_0$; (P3) construct a confidence set for γ; (P4) test H_0': $\gamma = \gamma_0$.

Given the true autocorrelation coefficient ρ, the standard statistic to test $\tilde{H}_0$: $\gamma = \gamma_0$ is the corresponding Fisher statistic $F(\gamma_0, \rho)$ based on the transformed model (3.6):

$$(3.11) \qquad F(\gamma_0, \rho) = [\hat{\gamma}(\rho) - \gamma_0]'\{C[X(\rho)'X(\rho)]^{-1}C'\}^{-1}$$
$$\times [\hat{\gamma}(\rho) - \gamma_0]/[ms(\rho)^2],$$

[3] If we assumed that u_1 is random with mean zero, β_1 would be identified even if $\rho = 1$, and β_1 could be estimated by $\hat{\beta}_1 = y_1 - z_1'\hat{\gamma}$, where $\hat{\gamma}$ is the least-squares estimate of $\bar{\gamma}$ obtained from (3.5) with $\rho = 1$. However, the latter estimate depends very heavily on a single observation (y_1) and, unless an extra assumption is imposed on the variance of u_1, the variance of $\hat{\beta}_1$ cannot be estimated when $\rho = 1$. When $\text{Var}(u_1)$ is a free parameter and $\rho = 1$, it is not possible to build confidence intervals for β_1. A possible assumption that would allow one to do this is $\text{Var}(u_1) = \sigma^2$ (Berenblut and Webb (1973)).

where $\hat{\gamma}(\rho) = C\hat{\beta}(\rho)$.[4] Clearly, $F(\gamma, \rho)$ follows a Fisher distribution with $(m, T_1 - k)$ degrees of freedom, for γ and ρ are the true parameter values. The test rejects $\tilde{H}_0$ at level α when $F(\gamma_0, \rho) > \tilde{F}(\alpha)$, where $\tilde{F}(\alpha) \equiv F(\alpha; m, T_1 - k)$, $P[F(m, T_1 - k) > F(\alpha; m, T_1 - k)] = \alpha$ and $0 \leqslant \alpha \leqslant 1$; we set $\tilde{F}(0) = +\infty$ and $\tilde{F}(1) = 0$. Correspondingly, we get a confidence set for γ with level $1 - \alpha$ by considering the set of values γ_0 that are not rejected by the test:

$$(3.12) \qquad J(\alpha, \rho) = \left\{\gamma_0 \colon F(\gamma_0, \rho) \leqslant \tilde{F}(\alpha)\right\}.$$

By construction, $P[\gamma \in J(\alpha, \rho)] = 1 - \alpha$.

Suppose now that ρ is unknown but we can find an exact confidence set for ρ with level $1 - \alpha_1$: for all admissible values of ρ,

$$P\left[\rho \in I(\alpha_1)\right] = 1 - \alpha_1,$$

or, more generally,

$$(3.13) \qquad P\left[\rho \in I(\alpha_1)\right] \geqslant 1 - \alpha_1,$$

where $0 \leqslant \alpha_1 \leqslant 1$ and $I(\alpha_1) = I(Y; \alpha_1)$ is a random set determined by Y. As we will see below, a set that satisfies (3.13), i.e. a conservative confidence set, is sufficient for our purposes. Further, we will make the following assumption:

ASSUMPTION C: *The event* $\{\rho \in I(\alpha_1)\}$ *and the random variable* $F(\gamma, \rho)$ *are independent.*

In Section 4, we will see that most reasonable methods of building confidence sets for ρ satisfy this assumption. Further, it is easy to relax it (see the end of this section).

Let $0 \leqslant \alpha_1 \leqslant 1$, $0 \leqslant \alpha_2 \leqslant 1$, and consider the set

$$(3.14) \qquad \begin{aligned} K(\alpha_1, \alpha_2) &= \left\{(\rho_0, \gamma_0) \colon \rho_0 \in I(\alpha_1) \text{ and } \gamma_0 \in J(\alpha_2, \rho_0)\right\} \\ &= \left\{(\rho_0, \gamma_0) \colon \rho_0 \in I(\alpha_1) \text{ and } F(\gamma_0, \rho_0) \leqslant \tilde{F}(\alpha_2)\right\}. \end{aligned}$$

In general, we have

$$(3.15) \qquad \begin{aligned} \bar{P}(\alpha_1, \alpha_2) &\equiv P\left[(\rho, \gamma) \in K(\alpha_1, \alpha_2)\right] = P\left[\rho \in I(\alpha_1) \text{ and } \gamma \in J(\alpha_2, \rho)\right] \\ &= P\left[\rho \in I(\alpha_1) \text{ and } F(\gamma, \rho) \leqslant \tilde{F}(\alpha_2)\right]. \end{aligned}$$

If Assumption C holds and $P[\rho \in I(\alpha_1)] = 1 - \alpha_1$, then

$$(3.16) \qquad \bar{P}(\alpha_1, \alpha_2) = P\left[\rho \in I(\alpha_1)\right] P\left[F(\gamma, \rho) \leqslant \tilde{F}(\alpha_2)\right] = (1 - \alpha_1)(1 - \alpha_2).$$

Thus $K(\alpha_1, \alpha_2)$ is a similar confidence set for (ρ, γ) with level $(1 - \alpha) \equiv$

[4] Since there is no uniformly most powerful test of $\gamma = \gamma_0$ against $\gamma \neq \gamma_0$, the Fisher test is optimal only in limited classes of tests, e.g. tests that obey certain invariance properties. For a general discussion, see Scheffé (1959, Chapter 2). If further restrictions are imposed on the alternative (e.g., $\gamma \geqslant 0$), more powerful tests could be obtained; for an example, see King and Smith (1986). Note, however, that the approach developed in this paper could also be applied to such problems.

$(1-\alpha_1)(1-\alpha_2)$. By selecting α_1 and α_2 appropriately, we can get any desired confidence level; for example, we may take $\alpha_1 = \alpha_2 = 1-(1-\alpha)^{1/2}$. Moreover, the confidence set $K(\alpha_1, \alpha_2)$ is conservative (liberal) when $I(\alpha_1)$ is conservative (liberal):

$$(3.17)\quad \begin{aligned} \overline{P}(\alpha_1,\alpha_2) &\geqslant (1-\alpha_1)(1-\alpha_2), \quad \text{if} \quad P[\rho \in I(\alpha_1)] \geqslant (1-\alpha_1), \\ &\leqslant (1-\alpha_1)(1-\alpha_2), \quad \text{if} \quad P[\rho \in I(\alpha_1)] \leqslant (1-\alpha_1). \end{aligned}$$

Given the joint confidence set $K(\alpha_1, \alpha_2)$, it is straightforward to test H_0: $\rho = \rho_0$, $\gamma = \gamma_0$: one simply rejects H_0 when $(\rho_0, \gamma_0) \notin K(\alpha_1, \alpha_2)$. Clearly

$$(3.18)\quad \begin{aligned} P[(\rho,\gamma) \notin K(\alpha_1,\alpha_2)] = 1-\overline{P}(\alpha_1,\alpha_2) &= \alpha, \quad \text{if} \quad \overline{P}(\alpha_1,\alpha_2) = 1-\alpha, \\ &\leqslant \alpha, \quad \text{if} \quad \overline{P}(\alpha_1,\alpha_2) \geqslant 1-\alpha. \end{aligned}$$

We now consider the problem of making inferences about γ only. As above, it will be convenient to deal first with the construction of a confidence set for γ. For this purpose, we define the sets

$$(3.19)\quad U(\alpha_1,\alpha_2) = \{\gamma_0 \colon (\rho_0,\gamma_0) \in K(\alpha_1,\alpha_2) \text{ for some } \rho_0 \in I(\alpha_1)\},$$

$$(3.20)\quad L(\alpha_1,\alpha_2) = \{\gamma_0 \colon (\rho_0,\gamma_0) \in K(\alpha_1,\alpha_2) \text{ for all } \rho_0 \in I(\alpha_1)\},$$

and show the following proposition.

PROPOSITION 1: *Suppose that the model described by* (1.1), (3.9), *and either Assumption A or Assumption B holds. Let* $I(\alpha_1)$ *be a confidence set for* ρ *such that* $P[\rho \in I(\alpha_1)] \geqslant 1-\alpha_1$, *where* $0 \leqslant \alpha_1 \leqslant 1$, *and suppose that Assumption C holds. If* $0 < \alpha < 1$ *and* α_1, α_2 *and* α_2' *are chosen so that*

$$(3.21)\quad (1-\alpha_1)(1-\alpha_2) = 1-\alpha, \quad (1-\alpha_1)\alpha_2' = \alpha, \quad 0 \leqslant \alpha_1 < \alpha \leqslant 1-\alpha_1,$$

then $\alpha_2 \leqslant \alpha \leqslant \alpha_2'$, $L(\alpha_1,\alpha_2') \subseteq U(\alpha_1,\alpha_2)$, *and*

$$(3.22)\quad P[\gamma \in L(\alpha_1,\alpha_2')] \leqslant 1-\alpha \leqslant P[\gamma \in U(\alpha_1,\alpha_2)]$$

for all admissible values of ρ *and* γ.

The proof of this proposition is given in Appendix A.1. When (3.21) holds, Proposition 1 implies that $U(\alpha_1, \alpha_2)$ is a conservative confidence set for γ with level $1-\alpha$, while $L(\alpha_1, \alpha_2')$ is a liberal set with the same level.[5] It is easy to derive conservative and liberal tests from these confidence sets. For $0 < \alpha < 1$, choose α_1, α_2, and α_2' so that (3.21) holds and define

$$(3.23)\quad \begin{aligned} Q_L(\gamma) &= \operatorname{Inf}\{F(\gamma,\rho_0) \colon \rho_0 \in I(\alpha_1)\}, \\ Q_U(\gamma) &= \operatorname{Sup}\{F(\gamma,\rho_0) \colon \rho_0 \in I(\alpha_1)\}. \end{aligned}$$

[5] The confidence sets $U(\alpha_1, \alpha_2)$ and $L(\alpha_1, \alpha_2)$ also have a "Bayesian interpretation" that can be of interest: If, in model (3.1), the prior distribution of (β, σ^2) is diffuse of the form $p(\beta, \sigma^2) \propto 1/\sigma^2$ (see Zellner (1971, p. 66)) and ρ is independent of (β, σ^2), then $U(\alpha_1, \alpha_2)$ and $L(\alpha_1, \alpha_2)$ are respectively the union and the intersection of posterior confidence sets (with level $1-\alpha_2$) conditional on each value of ρ in $I(\alpha_1)$. This holds irrespective of the prior distribution on ρ.

By Proposition 1, $P[\gamma \in U(\alpha_1, \alpha_2)] \geqslant 1 - \alpha$ and $P[\gamma \in L(\alpha_1, \alpha_2')] \leqslant 1 - \alpha$. Further, the event $\gamma \notin U(\alpha_1, \alpha_2)$ is equivalent (with probability 1) to $Q_L(\gamma) \geqslant \tilde{F}(\alpha_2)$ while the event $\gamma \in L(\alpha_1, \alpha_2')$ is equivalent to $Q_U(\gamma) \leqslant \tilde{F}(\alpha_2')$. Thus

$$(3.24) \qquad P\left[Q_L(\gamma) \geqslant \tilde{F}(\alpha_2)\right] \leqslant \alpha, \quad P\left[Q_U(\gamma) \leqslant \tilde{F}(\alpha_2')\right] \leqslant 1 - \alpha.$$

$Q_L(\gamma_0) > \tilde{F}(\alpha_2)$ is a conservative critical region while $Q_U(\gamma_0) > \tilde{F}(\alpha_2')$ is a liberal critical region for testing $\gamma = \gamma_0$ at level α. Further, $Q_L(\gamma_0) \leqslant Q_U(\gamma_0)$ and $\tilde{F}(\alpha_2) \geqslant \tilde{F}(\alpha_2')$ so that the conservative critical region is contained in the liberal one. This suggests the following bounds test for H_0': $\gamma = \gamma_0$:

$$(3.25) \qquad \begin{cases} \text{reject } H_0' \text{ when } Q_L(\gamma_0) > \tilde{F}(\alpha_2), \\ \text{accept } H_0' \text{ when } Q_U(\gamma_0) \leqslant \tilde{F}(\alpha_2'), \\ \text{consider the test inconclusive, otherwise.} \end{cases}$$

Proposition 1 relies on the apparently restrictive independence Assumption C. The following proposition (also proved in the Appendix) relaxes this assumption.

PROPOSITION 2: *Suppose that the model described by* (1.1), (3.9), *and either Assumption A or Assumption B holds, and let* $I(\alpha_1)$ *be a confidence set for* ρ *such that* $P[\rho \in I(\alpha_1)] \geqslant 1 - \alpha_1$. *If* $0 < \alpha < 1$ *and* α_1, α_2, *and* α_2' *are chosen so that*

$$(3.26) \qquad \alpha_1 + \alpha_2 = \alpha, \quad \alpha_2' - \alpha_1 = \alpha, \quad 0 \leqslant \alpha_1 < \alpha \leqslant 1 - \alpha_1,$$

then the inequality (3.22) *holds for all admissible values of* ρ *and* γ.

Here, no assumption is made on the stochastic relationship between the event $\{\rho \in I(\alpha_1)\}$ and the random variable $F(\gamma, \rho)$. Clearly, the bounds test (3.25) remains exact when (3.26) holds. When α_1 and α_2 are small, as is the case in practice (e.g., when $\alpha = 0.05$ and $\alpha_1 = \alpha_2$), there is little difference between the critical bounds and confidence sets based on (3.26) and those based on (3.21); to see this, compare the inequalities (A.1)–(A.2) with (A.3)–(A.4) in the Appendix.

An especially simple case of the test in (3.25) is obtained by taking the set of all admissible values of ρ as the confidence set for ρ. Then $\alpha_1 = 0$, $\alpha_2 = \alpha_2' = \alpha$ and, under Assumption B, we have $I(\alpha_1) = I(0) = \{\rho_0\colon\ |\rho_0| \leqslant 1\}$ yielding the bounds test:

$$(3.27) \qquad \begin{cases} \text{reject } H_0' \text{ when } \operatorname{Inf}\{F(\gamma_0, \rho_0)\colon\ |\rho_0| \leqslant 1\} > \tilde{F}(\alpha), \\ \text{accept } H_0' \text{ when } \operatorname{Sup}\{F(\gamma_0, \rho_0)\colon\ |\rho_0| \leqslant 1\} \leqslant \tilde{F}(\alpha), \\ \text{consider the test inconclusive, otherwise.} \end{cases}$$

Note that the conditions (3.21) and (3.26) both hold in this case. An analogous test is also available under Assumption A ($|\rho| < 1$). This procedure does not require computing a confidence interval for ρ from the available sample. On the other hand, it does not exploit the information contained in the data about ρ. This can lead to unduly large inconclusive regions. For this reason, we now study how to construct an exact confidence set for ρ.

4. CONFIDENCE SETS FOR THE AUTOCORRELATION COEFFICIENT

Suppose first that we wish to test the hypothesis $H(\rho_0)$: $\rho = \rho_0$. Under $H(\rho_0)$, it is appropriate to consider the transformed model

$$y(\rho_0) = X(\rho_0)\tilde{\beta} + u(\rho_0), \tag{4.1}$$

as given by (3.6) to (3.9). When $H(\rho_0)$ is true, the disturbances $u_t(\rho_0)$ of the transformed model are independent $N(0, \sigma^2)$. If $\rho \neq \rho_0$, they remain autocorrelated and follow an ARMA(1,1) process:

$$u_t(\rho_0) - \rho u_{t-1}(\rho_0) = \varepsilon_t - \rho_0 \varepsilon_{t-1}. \tag{4.2}$$

Thus, in particular, the sequence $u_t(\rho_0)$, $t = p, \ldots, T$, is autocorrelated at lag 1. We can test $H(\rho_0)$ by testing the equivalent hypothesis that the residuals of the transformed model are independent. Any test having reasonable power against the alternative (4.2) is a possible choice here though, of course, powers may differ between alternative tests. Note also that for the special case $\rho_0 = 1$, more specific procedures have been developed; see Evans and Savin (1981, 1984), Sargan and Bhargava (1983), and Bhargava (1986). However, to obtain a confidence set for ρ, we need a general procedure to allow testing any admissible value ρ_0.

Consider any statistic of the form

$$d(\rho_0) = \hat{u}(\rho_0)' A \hat{u}(\rho_0) / \hat{u}(\rho_0)' \hat{u}(\rho_0) \tag{4.3}$$

where A is a fixed matrix (possibly a function of ρ_0). For example, we may consider the Durbin-Watson (DW) statistic

$$DW(\rho_0) = \sum_{t=p}^{T-1} [\hat{u}_{t+1}(\rho_0) - \hat{u}_t(\rho_0)]^2 \Big/ \sum_{t=p}^{T} \hat{u}_t(\rho_0)^2, \tag{4.4}$$

which is employed usually to test independence against autocorrelation at lag 1. In this case, the matrix A does not depend on ρ_0. Most variants of the Durbin-Watson statistic have the form (4.3); see Kadiyala (1970) and King (1987). Typically one rejects the null hypothesis of independence when the test statistic is too small or too large:

$$d(\rho_0) < d_1(\rho_0, X) \quad \text{or} \quad d(\rho_0) > d_2(\rho_0, X). \tag{4.5}$$

$d_1(\rho_0, X)$ and $d_2(\rho_0, X)$ are selected so that, under $H(\rho_0)$,

$$P[d(\rho_0) < d_1(\rho_0, X)] = \delta_1, \quad P[d(\rho_0) > d_2(\rho_0, X)] = \delta_2, \tag{4.6}$$

where $\delta_1 + \delta_2 = \alpha$, $0 < \alpha < 1$, $0 \leqslant \delta_i \leqslant 1$, $i = 1, 2$. In general, d_1 and d_2 depend on X and ρ_0, though not always. For example, the DW statistic has a null distribution that depends upon the matrix $X(\rho_0)$ while tests based on LUS (linear uncorrelated scalar) residuals, like BLUS or recursive residuals, do not.

In view of the possible dependence of critical values upon X and ρ_0, it is often convenient to use the cumulative distribution function of $d(\rho_0)$ under $H(\rho_0)$:

$$F(x; \rho_0, X) = P[d(\rho_0) \leqslant x | H(\rho_0)]. \tag{4.7}$$

If we set $\delta(\rho_0) = F[\hat{d}(\rho_0); \rho_0, X]$, where $\hat{d}(\rho_0)$ is the observed value of $d(\rho_0)$, the

test given in (4.5) is equivalent to rejecting $H(\rho_0)$ when

$$(4.8) \qquad \delta(\rho_0) < \delta_1 \quad \text{or} \quad 1 - \delta(\rho_0) < \delta_2.$$

We accept $H(\rho_0)$ when $d_1(\rho_0, X) \leqslant d(\rho_0) \leqslant d_2(\rho_0, X)$ or, equivalently, when $\delta_1 \leqslant \delta(\rho_0) \leqslant 1 - \delta_2$. We can thus obtain an exact confidence set for ρ with level $1 - \alpha$ by finding the set of admissible values of ρ that are not rejected by the test:

$$(4.9) \qquad \begin{aligned} I(\alpha) &= \{\rho_0 \in S\colon\ d_1(\rho_0, X) \leqslant d(\rho_0) \leqslant d_2(\rho_0, X)\} \\ &= \{\rho_0 \in S\colon\ \delta_1 \leqslant \delta(\rho_0) \leqslant 1 - \delta_2\} \end{aligned}$$

where $S = \{\rho\colon\ |\rho| < 1\}$ or $S = \{\rho\colon\ |\rho| \leqslant 1\}$ (depending on the assumption adopted: A or B). It is straightforward to see that $P[\rho \in I(\alpha)] = 1 - \alpha$. Moreover if, instead of (4.6), we have the inequalities

$$(4.10) \qquad P[d(\rho_0) < d_1] \leqslant \delta_1, \qquad P[d(\rho_0) > d_2] \leqslant \delta_2,$$

the set $I(\alpha) = \{\rho_0\colon\ d_1 \leqslant d(\rho_0) \leqslant d_2\}$ is a conservative confidence set with level $1 - \alpha$, i.e. $P[\rho \in I(\alpha)] \geqslant 1 - \alpha$. Note also that the restrictions $|\rho| < 1$ and $|\rho| \leqslant 1$ make sure that the confidence set $I(\alpha)$ is bounded.

When the null distribution of $d(\rho_0)$ does not depend on the matrix $X(\rho_0)$, d_1 and d_2 only depend on the level of the test. For example, if we apply the von Neumann ratio (or the modified von Neumann ratio) to a set of LUS residuals, we can use tables for the von Neumann ratio to obtain the critical values d_1 and d_2 (provided, of course, the desired level is available). When $|\rho| < 1$ or $|\rho| \leqslant 1$, the interval may easily be identified by performing a grid search over the admissible interval and by retaining all values ρ_0 such that $d_1 \leqslant d(\rho_0) \leqslant d_2$.

When no table is available, several algorithms may be used to compute $\delta(\rho_0)$ and search for those values that satisfy $\delta_1 \leqslant \delta(\rho_0) \leqslant 1 - \delta_2$; see Imhof (1961), Koerts and Abrahamse (1969), Pan Jie-Jian (1964), Davies (1980), Palm and Sneek (1984), Farebrother (1984, 1985). We may also determine approximately the limits of the confidence set by using an approximate but inexpensive method of computing the distribution of a ratio of two quadratic forms in normal variables (see Durbin and Watson (1971), Evans and King (1985)); afterwards, we can use a more expensive algorithm to determine precisely these limits. Furthermore, in some important cases, it is possible to combine the use of a table with Imhof's algorithm or with an equivalent method.

Let us consider in detail the case where the DW statistic is used. Take $\delta_1 = \delta_2 = \alpha/2$. From DW tables, we can typically find points d_{1L}, d_{1U}, d_{2L}, and d_{2U} which do not depend on the regressor matrix and such that

$$P[DW(\rho_0) < d_{1L}] \leqslant \frac{\alpha}{2} \leqslant P[DW(\rho_0) < d_{1U}],$$

and

$$P[DW(\rho_0) < d_{2L}] \leqslant 1 - \frac{\alpha}{2} \leqslant P[DW(\rho_0) < d_{2U}],$$

provided the model contains an intercept (see Durbin and Watson (1950)). Then for all values of X and ρ_0, we have

$$d_{1L} \leqslant d_1(X, \rho_0) \leqslant d_{1U}, \qquad d_{2L} \leqslant d_2(X, \rho_0) \leqslant d_{2U}.$$

Let

$$I_1 = \{\rho_0 \colon d_{1L} \leqslant DW(\rho_0) \leqslant d_{2U}\}, \qquad I_2 = \{\rho_0 \colon d_{1U} \leqslant DW(\rho_0) \leqslant d_{2L}\}$$

and

$$I(\alpha) = \{\rho_0 \colon d_1(X, \rho_0) \leqslant DW(\rho_0) \leqslant d_2(X, \rho_0)\}.$$

Then $I_2 \subseteq I(\alpha) \subseteq I_1$; hence

$$P[\rho \in I_2] \leqslant P[\rho \in I(\alpha)] = 1 - \alpha \leqslant P[\rho \in I_1].$$

I_1 is a conservative confidence set (with level $1-\alpha$) while I_2 is a liberal confidence set. Since it is simple to compute $DW(\rho_0)$, we can find easily the two sets I_1 and I_2. To determine $I(\alpha)$ more tightly, it is then sufficient to search in the smaller set $J = I_1 \setminus I_2$ by a more precise method.

We will now use the following proposition.

PROPOSITION 3: *Let $y = X\beta + u$ and $u \sim N[0, \sigma^2 I_T]$, where X is a $T \times k$ fixed matrix with* $\operatorname{rank}(X) = k < T$ *and β is a vector of coefficients, and let*

$$\hat{\beta} = (X'X)^{-1}X'y, \qquad \hat{u} = y - X\hat{\beta}, \qquad \|\hat{u}\|^2 = \hat{u}'\hat{u}.$$

Then $\hat{\beta}$, $\|\hat{u}\|$, and $\hat{u}/\|\hat{u}\|$ are mutually independent.

This proposition can be proved easily by using Basu's (1966) theorem or by transforming $\hat{u}$ to polar coordinates; for a detailed proof based on the latter approach, see Dufour (1986).[6] Since the transformed model described by (3.6) to (3.9) satisfies the assumptions of the classical linear model, we can conclude from Proposition 3 that $\hat{\beta}(\rho)$, $\|\hat{u}(\rho)\|$, and $\hat{u}(\rho)/\|\hat{u}(\rho)\|$ are independent. Further, the test statistic $d(\rho)$, as given by (4.3), is a function of $\hat{u}(\rho)/\|\hat{u}(\rho)\|$ while the Fisher statistic $F(\gamma, \rho)$ defined by (3.11) depends on $\hat{\beta}(\rho)$ and $\|\hat{u}(\rho)\|$ only, so that $d(\rho)$ and $F(\gamma, \rho)$ are independent. Thus the event $\rho \in I(\alpha)$, where $I(\alpha)$ is defined by (4.9), is independent of $F(\gamma, \rho)$. To obtain an exact test of $\gamma = \gamma_0$, we can use Proposition 1 with $(1-\alpha_1)(1-\alpha_2) = 1-\alpha$ and $(1-\alpha_1)\alpha_2' = \alpha$.

5. NUMERICAL EXAMPLES

To illustrate the methods described above, we now give two examples based on artificial data. In both cases, the equation estimated has the form

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t \qquad (t = 1, \ldots, T), \tag{5.1}$$

[6] Though various implications or alternative forms of this proposition were used by several authors (see Geary (1933), Pitman (1937), von Neumann (1941), Durbin and Watson (1950), Kariya and Eaton (1977), Kendall and Stuart (1977, p. 287), Phillips and McCabe (1985)), we did not find a complete statement of it elsewhere.

and the sample size is 50 ($T = 50$). In the first example, x_{2t} and x_{3t} were generated by processes of the form

$$x_{2t} = 0.5x_{2,t-1} + \eta_{2t}, \qquad x_{2,0} = \eta_{2,0}/\sqrt{0.75}, \tag{5.2}$$

$$x_{3t} = t + \eta_{3t}, \tag{5.3}$$

where $\eta_{2t} \sim N[0,1]$, $\eta_{3t} \sim N[0,10]$, and all η_{2t}, η_{3t}, $t = 0,1,\ldots,T$, are mutually independent. We obtained y_t by using equation (5.1) with $\beta_1 = 10$, $\beta_2 = \beta_3 = 1$, and

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{ind}}{\sim} N[0,\sigma^2] \qquad (t = 2,\ldots,T), \tag{5.4}$$

$$u_1 \sim N\left[0, \sigma^2/(1-\rho^2)\right],$$

where $\rho = 0.9$, $\sigma = 2$, and ε_t is independent of u_1 and (x_{2s}, x_{3s}), $s = 1,\ldots,T$. The second data set is identical to the first one except that the equation generating y_t had $\rho = -0.9$ and x_{3t} was divided by 10.

When equation (5.1) is estimated by the Hildreth-Lu grid search algorithm, the results obtained for each of the data sets are respectively:

$$\underset{}{y_t =} \underset{(2.993)}{10.360} + \underset{(0.268)}{0.766}\, x_{2t} + \underset{(0.0646)}{0.991}\, x_{3t} + \hat{u}_t, \qquad \underset{}{\hat{\rho} =} \underset{(0.070)}{0.87},$$

$$SS = 174.4, \quad SE = 1.947, \quad DW = 1.67, \quad R^2 = 0.837;$$

$$\underset{}{y_t =} \underset{(0.234)}{9.938} + \underset{(0.098)}{0.884}\, x_{2t} + \underset{(0.081)}{1.069}\, x_{3t} + \hat{u}_t, \qquad \underset{}{\hat{\rho} =} \underset{(0.029)}{-0.98},$$

$$SS = 108.8, \quad SE = 1.537, \quad DW = 2.11, \quad R^2 = 0.865.$$

Standard errors are given in parentheses; $\hat{\rho}$ is the final estimated value of the autoregressive coefficient, SS is the sum of squared residuals, SE is the standard error of the regression, DW is the Durbin-Watson statistic, R^2 is the coefficient of determination. Standard errors are conditional standard errors.[7]

The results of applying the methods described in the previous sections to these data are reported in Table I. We consider three ways of obtaining a confidence set for ρ, all valid under the assumption $|\rho| \leqslant 1$ (Assumption B). For the two first methods, we take $\alpha_1 = \alpha_2 = 1 - (1-\alpha)^{1/2}$ and get a confidence set for ρ with level $1 - \alpha_1$ (at least) by determining the set of values $|\rho_0| \leqslant 1$ such that the Durbin-Watson statistic of the transformed model is not significant at level α_1. In the first method, critical levels are obtained by using the Imhof (1961) algorithm as implemented by Koerts and Abrahamse (1969): this yields a confidence set for ρ with level $1 - \alpha_1$ (tight). In the second method, we decide which values ρ_0 are acceptable by using a conservative bound obtained from a standard bounds table: this yields a *conservative* confidence set for ρ with level $1 - \alpha_1$, without the need to use Imhof's algorithm. In the third method, we use the complete set of

[7] The data are available from the author upon request. We also estimated the models by the Cochrane-Orcutt algorithm and by the Beach-MacKinnon (1978) maximum likelihood algorithm. The results are very close to those presented here.

admissible values $|\rho_0| \leqslant 1$ as a confidence set with level 1, i.e. $\alpha_1 = 0$ and $\sigma_2 = \alpha$. In all the examples, the basic confidence level is 0.95 ($\alpha = 0.05$). We will concentrate our discussion on the first data set ($\rho = 0.90$).

Let us first look at the results based on constructing a tight confidence set for ρ with level $1 - \alpha_1 = 0.9747$. In this example, $T = 50$, $T_1 = 49$, and $k = 3$. The interval for ρ is then $0.69 \leqslant \rho \leqslant 1.00$. We can see that the (conservative) confidence intervals for $\tilde{\beta}_1$, β_2, and β_3 are all reasonably small. As expected, the confidence intervals for the regression coefficients β_2 and β_3 are larger than the asymptotic confidence intervals. The latter are $0.241 \leqslant \beta_2 \leqslant 1.291$ and $0.864 \leqslant \beta_3 \leqslant 1.118$. It is clear that β_2 and β_3 are significantly different from zero: the bounds tests for $\beta_2 = 0$ and $\beta_3 = 0$ are significant at level 0.05 because

$$Q_L(\beta_2 = 0) = (2.85)^2 > (2.31)^2 = F(\alpha_2; 1, 46),$$

$$Q_L(\beta_3 = 0) = (14.7)^2 > (2.31)^2 = F(\alpha_2; 1, 46),$$

where $\alpha_2 = 0.0253$. On the other hand, the test for $\tilde{\beta}_1 = 0$ is inconclusive because $Q_L(\tilde{\beta}_1 = 0) = 0.31 < F(\alpha_2; 1, 46)$ and $Q_U(\tilde{\beta}_1 = 0) = 4.39 > F(\alpha_2'; 1, 46)$. Given that $\rho = 1$ (a value contained in the confidence set for ρ) implies that $\tilde{\beta}_1 = (1 - \rho)\beta_1 = 0$, this is not surprising. Note also that a confidence interval for β_1 is not reported because β_1 is not identified in the transformed model when $\rho = 1$.

The second set of results is based on finding the values ρ_0 that satisfy $d_L \leqslant DW(\rho_0) \leqslant 4 - d_L$, where $d_L = 1.25$ is the lower critical bound (a uniformly conservative critical value) of the DW test with level 0.01 against positive autocorrelation. This critical value is available from standard tables; see Savin and White (1977, Table II). The confidence set for ρ obtained in this way is conservative at level 0.975.[8] By construction, this must lead to larger confidence sets but no use of the Imhof algorithm is required. The ρ-interval obtained in this way is [0.60, 1.00], which is close to the result obtained by the first method. Correspondingly, the confidence intervals and test statistics for the regression coefficients are very similar to those obtained by the first method.

In contrast, the results based on using the full interval $|\rho| \leqslant 1$ ($\alpha_1 = 0$, $\alpha_2 = 0.05$) yield much wider confidence intervals for $\tilde{\beta}_1$ and β_2. This brings support to the presumption that finding a confidence set for ρ yields more accurate results in most situations. On the other hand, the confidence interval for β_3 has practically the same length as with the previous method. The bounds tests for $\beta_2 = 0$ and $\beta_3 = 0$ are significant at level 0.05. Finally, it is interesting to note that the bounds test of Vinod (1976) and Kiviet (1980) is inconclusive for $\beta_2 = 0$: the OLS t statistic is 3.243 while the upper bound with $\rho = 0.9$ is 12.65; see Kiviet (1980, Table 4). The highest ρ value considered by Kiviet (1980) is $\rho = 0.9$.

The main features of the results obtained from the second data set ($\rho = -0.90$) are similar to those of the first example. Note, however, that the confidence intervals for ρ and β_2 are considerably shorter. Furthermore, since 1 is not

[8] In fact, the confidence level of the interval for ρ is not smaller than $1 - 2(.01) = .98$. A somewhat shorter interval could be obtained by using the lower critical bound associated to a (one-sided) test of level $0.0253/2 = 0.01265$. However, this value is not tabulated (though it could be found easily). Clearly, the results would be very close to those presented here.

TABLE I

NUMERICAL EXAMPLES ($\alpha = 0.05$)[a]

	Confidence intervals				
	ρ	β_1	$\tilde{\beta}_1$	β_2	β_3
First data set ($\rho = 0.90$)					
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = \alpha_2 = 0.0253$	[0.69, 1.00]	—	[−0.590, 3.734]	[0.147, 1.541]	[0.823, 1.191]
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 \leqslant 0.0253$, $\alpha_2 = 0.0253$[b]	[0.60, 1.00]	—	[−0.590, 4.271]	[0.147, 1.643]	[0.823, 1.208]
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = 0$, $\alpha_2 = \alpha_2' = 0.05$	[−1.00, 1.00]	—	[−0.501, 15.20]	[0.227, 2.146]	[0.843, 1.219]
Second data set ($\rho = -0.90$)					
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = \alpha_2 = 0.0253$	[−1.00, −0.81]	[9.299, 10.668]	[16,832, 20.939]	[0.596, 1.165]	[0.813, 1.288]
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 \leqslant 0.0253$, $\alpha_2 = 0.0253$[b]	[−1.00, −0.79]	[9.275, 10.702]	[16.603, 20.939]	[0.584, 1.177]	[0.801, 1.295]
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = 0$, $\alpha_2 = \alpha_2' = 0.05$	[−1.00, 1.00]	—	[−5.559, 20.801]	[−0.462, 5.179]	[−1.033, 15.543]

	Test statistics for zero regression coefficients									
	β_1		$\tilde{\beta}_1$		β_2		β_3		$\sqrt{\tilde{F}(\alpha_2)}$	$\sqrt{\tilde{F}(\alpha_2')}$
	$\lvert t_L\rvert$	$\lvert t_U\rvert$	$\lvert t_L\rvert$	$\lvert t_U\rvert$	$\lvert t_L\rvert$	$\lvert t_U\rvert$	$\lvert t_L\rvert$	$\lvert t_U\rvert$		
First data set ($\rho = 0.90$)										
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = \alpha_2 = 0.0253$	—	—	0.31	4.39	2.85*	2.95	14.70*	19,61	2.31	2.00
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 \leqslant 0.0253$, $\alpha_2 = 0.0253^b$	—	—	0.31	4.56	2.85*	2.95	14.70*	22.26	2.31	2.00
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = 0$, $\alpha_2 = \alpha_2' = 0.05$	—	—	0.31	5.76	2.85*	3.25	14.70*	33.90	2.01	2.01
Second data set ($\rho = -0.90$)										
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = \alpha_2 = 0.0253$	33.74*	42.73	33.74*	42.73	7.16*	9.14	10.23*	13.27	2.31	2.00
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 \leqslant 0.0253$, $\alpha_2 = 0.0253^b$	32.38*	42.73	32.38*	42.73	6.87*	9.14	9.80*	13.27	2.31	2.00
$\lvert\rho\rvert \leqslant 1$, $\alpha_1 = 0$, $\alpha_2 = \alpha_2' = 0.05$	—	—	0	42.73	1.55	9.14	1.27	13.27	2.01	2.01

[a] The confidence intervals for ρ have level $1 - \alpha_1$. The confidence intervals for $\tilde{\beta}_1 = (1 - \rho)\beta_1$, β_1, β_2, and β_3 are conservative at level $1 - \alpha = 0.95$. The test statistics for testing whether each regression coefficient is significantly different from zero ($\beta_i = 0$) are expressed in terms of the corresponding t statistics (instead of F statistics): $\lvert t_L\rvert = \sqrt{Q_L}$ and $\lvert t_U\rvert = \sqrt{Q_U}$. For the two examples, $T_1 = 49$ and $k = 3$. * indicates a significant statistic (at level 0.05).

[b] These results are based on the acceptance region $d_L \leqslant DW(\rho) \leqslant 4 - d_L$ for ρ, where d_L is the conservative critical value of the Durbin-Watson test of level 0.01 against positive autocorrelation ($d_L = 1.245$ for $T_1 = 49$ and $k = 3$). The confidence interval for ρ so obtained has level not smaller than $1 - 2(0.01) = 0.98 \geqslant 1 - 0.0253$.

contained in the confidence intervals for ρ (except when the third method is used), we can build finite confidence intervals for β_1. The intervals for β_1 are all reasonably short.

The above examples show clearly that the approach to inference suggested in this paper can be implemented and yield definite conclusions. In most cases, it appears advantageous to obtain first a confidence interval for ρ instead of using the full set of admissible values $\lvert\rho\rvert \leqslant 1$ (under Assumption B). But even the latter method can yield conclusive tests as found in the above examples. Furthermore, when the DW statistic is used to construct a confidence interval for ρ, it is possible to get reasonably short intervals by using critical bounds from standard tables.

6. CONCLUSION

The approach suggested in this paper defines a class of inference procedures rather than a single procedure. For α given, one gets a different procedure for each way of selecting α_1 (which in turn determines α_2) and for each way of building a confidence set for ρ. It is important to note that α should be selected *a priori*, not on the basis of the results yielded by different choices of α_1 for a given sample. Clearly, different methods of selecting α_1 may lead to procedures with different power characteristics. Further research is needed to assess the properties of different ways of "sharing" the overall significance level between α_1 and α_2. However, it is unlikely that any particular choice dominates uniformly the others: in general, we can expect that the choice depends on the true (and unknown) parameter values. In the absence of further indications, we suggest to share equally the significance level ($\alpha_1 = \alpha_2$). This is the rule typically employed in the literature on simultaneous inference (e.g., in Bonferroni-type procedures) and test combination; see Miller (1981), Savin (1984), and Phillips and McCabe (1989).

A second element of choice is the test statistic which is "inverted" to build a confidence set for ρ. Applying the Durbin-Watson test to the transformed model has the advantage of being relatively simple and allows one to use widely available tables and programs. On the other hand, it does not have any known optimality property (except for testing $\rho = 0$). Though the results obtained in the examples studied were reasonable, it seems likely that smaller confidence sets could be obtained by using better tests of the hypothesis $\rho = \rho_0$ (e.g., optimal invariant tests). This is the topic of ongoing research. Note, however, that uniformly most powerful tests do not exist for autocorrelation hypotheses, except in very special situations (see Anderson (1948)): this suggests that no method can uniformly dominate all the others.

Finally, it is easy to see that the general approach used in this paper can be adapted to other situations, e.g., linear regressions with MA(1) or heteroskedastic disturbances. Whenever the covariance matrix of the disturbances depends on a nuisance parameter θ_1 (besides the scale coefficient σ^2), one can use a similar three-stage confidence approach to get exact tests and confidence sets, provided it is possible to build an exact confidence set for θ_1. Of course, when θ_1 has dimension greater than one, a multivariate confidence set for θ_1 is needed and computational problems are more important. More generally, if the parameters of a model can be reduced to a vector of nuisance parameters θ_1 and a vector of parameters of interest θ_2, the method suggested can be applied when two elements are present: first, it is possible to build an exact confidence set for θ_1; second, if the *true* value of θ_1 is known, it is possible to obtain an exact confidence set for θ_2. Once the confidence set for θ_1 is available, the two elements are combined to get a simultaneous confidence set for θ_1 and θ_2. Then, a union-intersection method is used to obtain confidence sets as well as tests for θ_2 that are valid irrespective of the true value of θ_1. The extension to other problems of the approach used in this paper is also the topic of ongoing research.

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APPENDIX

A.1. PROOF OF PROPOSITION 1: By definition, $(\rho, \gamma) \in K(\alpha_1, \alpha_2)$ implies that $(\rho_0, \gamma) \in K(\alpha_1, \alpha_2)$ for some $\rho_0 \in I(\alpha_1)$, hence

$$P[\gamma \in U(\alpha_1, \alpha_2)] = P[(\rho_0, \gamma) \in K(\alpha_1, \alpha_2) \text{ for some } \rho_0 \in I(\alpha_1)]$$
$$\geqslant P[(\rho, \gamma) \in K(\alpha_1, \alpha_2)]$$

and, by (3.17),

$$P[\gamma \in U(\alpha_1, \alpha_2)] \geqslant (1 - \alpha_1)(1 - \alpha_2). \tag{A.1}$$

Let $\bar{L}(\alpha_1, \alpha_2)$ be the complement of $L(\alpha_1, \alpha_2)$ in the space of admissible values for γ and let

$$L_1(\alpha_1, \alpha_2) = \{\gamma_0 \colon F(\gamma_0, \rho_0) > \tilde{F}(\alpha_2) \text{ for some } \rho_0 \in I(\alpha_1)\},$$
$$K_1(\alpha_1, \alpha_2) = \{(\rho_0, \gamma_0) \colon F(\gamma_0, \rho_0) > \tilde{F}(\alpha_2) \text{ and } \rho_0 \in I(\alpha_1)\}.$$

Clearly, $L_1(\alpha_1, \alpha_2) \subseteq \bar{L}(\alpha_1, \alpha_2)$,

$$L_1(\alpha_1, \alpha_2) = \{\gamma_0 \colon (\rho_0, \gamma_0) \in K_1(\alpha_1, \alpha_2) \text{ for some } \rho_0\},$$

and $(\rho, \gamma) \in K_1(\alpha_1, \alpha_2)$ implies $\gamma \in L_1(\alpha_1, \alpha_2)$. Then

$$P[\gamma \in \bar{L}(\alpha_1, \alpha_2)] \geqslant P[\gamma \in L_1(\alpha_1, \alpha_2)] \geqslant P[(\rho, \gamma) \in K_1(\alpha_1, \alpha_2)].$$

Further

$$P[(\rho, \gamma) \in K_1(\alpha_1, \alpha_2)] = P[\rho \in I(\alpha_1) \text{ and } F(\gamma, \rho) > \tilde{F}(\alpha_2)].$$

If $\{\rho \in I(\alpha_1)\}$ and $F(\gamma, \rho)$ are independent (Assumption C), we have

$$P[(\rho, \gamma) \in K_1(\alpha_1, \alpha_2)] = P[\rho \in I(\alpha_1)] P[F(\gamma, \rho) > \tilde{F}(\alpha_2)] \geqslant (1 - \alpha_1)\alpha_2.$$

Consequently,

$$P[\gamma \in L(\alpha_1, \alpha_2)] = 1 - P[\gamma \in \bar{L}(\alpha_1, \alpha_2)] \leqslant 1 - P[(\rho, \gamma) \in K_1(\alpha_1, \alpha_2)]$$
$$\leqslant 1 - (1 - \alpha_1)\alpha_2. \tag{A.2}$$

When $0 < \alpha < 1$, $0 \leqslant \alpha_1 < \alpha \leqslant 1 - \alpha_1$, $(1 - \alpha_1)(1 - \alpha_2) = 1 - \alpha$, and $(1 - \alpha_1)\alpha_2' = \alpha$, we have

$$1 - \alpha_2' = 1 - \frac{\alpha}{1 - \alpha_1} \leqslant 1 - \alpha \leqslant \frac{1 - \alpha}{1 - \alpha_1} = 1 - \alpha_2,$$

so that $\alpha_2 \leqslant \alpha \leqslant \alpha_2'$. Thus $\tilde{F}(\alpha_2) \geqslant \tilde{F}(\alpha_2')$ and

$$L(\alpha_1, \alpha_2') = \{\gamma_0 \colon F(\gamma_0, \rho_0) \leqslant \tilde{F}(\alpha_2') \text{ for some } \rho_0 \in I(\alpha_1)\}$$
$$\subseteq \{\gamma_0 \colon F(\gamma_0, \rho_0) \leqslant \tilde{F}(\alpha_2) \text{ for all } \rho_0 \in I(\alpha_1)\}$$
$$\subseteq \{\gamma_0 \colon F(\gamma_0, \rho_0) \leqslant \tilde{F}(\alpha_2) \text{ for some } \rho_0 \in I(\alpha_1)\}$$
$$= U(\alpha_1, \alpha_2).$$

Using (A.1) and (A.2), we can conclude that $P[\gamma \in U(\alpha_1, \alpha_2)] \geqslant 1 - \alpha$ and $P[\gamma \in L(\alpha_1, \alpha_2')] \leqslant 1 - \alpha$.

Q.E.D.

A.2. PROOF OF PROPOSITION 2: The proof of Proposition 2 is analogous to that of Proposition 1, except that, without Assumption C, it is not clear that (A.1) and (A.2) hold. By using Bonferroni's inequality, however, the latter can be replaced by

$$(A.3) \qquad P[\gamma \in U(\alpha_1, \alpha_2)] \geqslant 1 - (\alpha_1 + \alpha_2),$$

and

$$(A.4) \qquad P[\gamma \in L(\alpha_1, \alpha_2)] \leqslant (1 - \alpha_2) + \alpha_1,$$

from which (3.22) follows whenever (3.26) holds. *Q.E.D.*

A.3. EXPLOSIVE PROCESSES: The methods proposed in this paper can be extended to possibly explosive processes by considering the following more general assumption.

ASSUMPTION B′: $-\infty < \rho < +\infty$ *and* u_1 *is fixed* (*or independent of* $\varepsilon_2, \ldots, \varepsilon_T$ *with an arbitrary distribution*).

Inference for the model described by (1.1) and (3.9) under Assumption B′ can proceed as under Assumption B using the transformed model (3.1) with $p = 2$. The main difference comes from the fact that the domain of admissible values for ρ is now unrestricted. Since the confidence set $I(\alpha)$ is defined by nonlinear inequalities on ρ, it is then possible that $I(\alpha)$ be unbounded with some nonzero probability. Further, the determination of the set $I(\alpha)$ is clearly more difficult, for a grid search cannot cover the full range of possible values. One needs to study analytically the shape of the function $d(\rho_0)$ in (4.3) or the corresponding p-value function. Different test statistics may have different behaviors. This topic goes beyond the scope of the present paper.

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