RANK TESTS FOR SERIAL CORRELATION*

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1. Introduction

In this paper, we show how a family of linear rank statistics can be used to test the randomness of a sequence of random variables under a simple symmetry assumption and, in particular, without the standard normality assumption.

Let $X_1, \ldots, X_n$ be a time series of random variables having (marginal) density functions (pdf's) symmetric about a common known median μ (but not necessarily identical in other respects). Without loss of generality, we can assume μ to be zero[1]. We want to test the null hypothesis (H_o) that $X_1, \ldots, X_n$ are mutually independent against the alternative that these variables are "positively (or negatively) serially dependent". There is however, a problem in defining precisely the latter notion in a nonparametric context, especially since we did not assume covariances to be finite. By the symmetry assumption, we can see easily that, under H_o,

$$\text{Med}(X_tX_{t+1}) = 0, \quad t = 1, \ldots, n-1, \qquad (1.1)$$

where $\text{Med}(X_tX_{t+1})$ refers to the median of X_tX_{t+1}. Then a simple way of defining "positive serial dependence" consists in saying that the medians of the variables X_tX_{t+1} are positive:

$$\text{Med}(X_tX_{t+1}) > 0, \quad t = 1, \ldots, n-1; \qquad (1.2)$$

similarly, "negative serial dependence" can be defined via negative medians[2]:

$$\text{Med}(X_tX_{t+1}) < 0, \quad t = 1, \ldots, n-1. \qquad (1.3)$$

Another concept of dependence we could also use is the concept of "positive (or negative) quadrant dependence" introduced by Lehmann (1966). Some details about this notion are given in the Appendix. For our purpose, let us just mention here that, under the symmetry assumption, (strict) "positive quadrant dependence" between X_t and X_{t+1} implies $\text{Med}(X_tX_{t+1}) > 0$; further, if second moments exist, we also have in this case $E(X_tX_{t+1}) > 0$ so that the alternative (1.2) can be reexpressed in terms of positive autocovariances (or autocorrelations):

$$E(X_tX_{t+1}) \quad 0, \quad t = 1, \ldots, n-1; \qquad (1.4)$$

and similarly in the case of "negative quadrant dependence". Thus, under wide conditions, the tests described in this paper may be viewed as tests for serial correlation .

A natural statistic to look at for such a test is the sequence of the products:

$$Z_t = X_tX_{t+1}, \quad t = 1, \ldots, n-1. \qquad (1.5)$$

We propose to test H_o by applying linear rank tests for symmetry about zero to the sequence $Z_1, \ldots, Z_{n-1}$. The tests obtained in this way are both easy to run and applicable against a wide variety of alternative hypotheses, such as the ones where (1.1) or (1.2) holds. The linear rank statistics used integrate information about both the signs of the observations and their sizes (via the ranks of the absolute values $|Z_t|$, $t = 1, \ldots, n-1$). Furthermore, as shown in this paper, the exact null distributions of the test statistics can be obtained easily and, in some standard cases, are already well tabulated.

These characteristics can be contrasted with those of a number of other nonparametric tests for randomness. The tests considered by Wallis and Moore (1941), Moore and Wallis (1943) and Stuart (1952) use only the signs of the first differences $X_{t+1} - X_t$, $t = 1, \ldots, n-1$. Similarly, the runs tests proposed by David (1947), Goodman (1958) and Granger (1963) are based on the sequence of the signs of the observations X_t, $t = 1, \ldots, n$. In relation to this last set of procedures, we show that one of the tests in the family considered here reduces to a test based on the number of runs in the same sign sequence (with the exact small sample null distribution being supplied while Goodman (1958) and Granger (1963) provide only a large sample null distribution; note also that David's test (1947) is done conditionally on the number of +'s and -'s in the sign sequence. The test of Wald and Wolfowitz (1943), based on computing a circular correlation coefficient, is similarly performed conditionally on the set of values taken by the observations; furthermore, it is not computationally very convenient.

With respect to applications, let us mention that the fact that not all economic variables follow Gaussian distributions has been appreciated, both empirically and theoretically, by a number of authors, including Mandelbrot (1963), Fama (1965), Praetz (1972) and Blattberg and Gonedes (1974); and, interestingly, these authors proposed mainly symmetric distributions as alternatives to the Gaussian.

The above results are easily generalized and similar (exact) tests can be applied to any sequence of the type: X_tX_{t+k}, $t = 1, \ldots, n-k$, where $k > 1$, so that the dependence between observations separated by lags greater than one can also be assessed. Besides, it is interesting to observe that the standard normal tests based on sample correlation coefficients, such as $r_k = \sum_{t=1}^{n-k} X_tX_{t+k} / \sum_{t=1}^{n} X_t^2$, typically use asymptotic null distributions[3].

The family of tests considered is described in section 2. The exact null distributions of the test statistics are obtained in section 3. In sections 4 and 5, special tests in the family are considered; in particular, we obtain an exact runs test for which critical points can be found using tables of the binomial distribution, and an analogue of the Wilcoxon signed-rank test for which tables built for this last test may be used. In section 6, the preceding results are generalized in order to be able to test the dependence between observations separated by lags greater than one. Finally, in section 7, we make some further remarks concerning the applicability and limitations of the tests proposed.

2. Description of the Tests

It is easy to see that, under H_o, each random variable $Z_t = X_tX_{t+1}$, $t = 1, \ldots, n-1$, has a continuous distribution symmetric about zero; hence

$$P[Z_t \geq 0] = P[Z_t \leq 0] = \tfrac{1}{2}, \quad t = 1, \ldots, n-1. \quad (2.1)$$

In a wide variety of cases, non-independence will break this pattern. In particular, under an alternative of positive (negative) serial dependence the medians of the variables Z_t, $t = 1, \ldots, n-1$, will be shifted towards the right (left). This clearly suggests to test whether the variables $Z_1, \ldots, Z_{n-1}$ have median zero.

Gokhale (1970) proposed to test the independence of two random variables X and Y (assumed

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to have distributions symmetric about zero) via a "distribution-free test of symmetry" about zero of the distribution of XY, using a sample of N independent observations $(X_1, Y_1), \ldots, (X_N, Y_N)$. He also studied sufficient conditions for the symmetry of XY about zero to be equivalent to the independence of X and Y. But, clearly, the couples $(X_1, X_2), (X_2, X_3), \ldots, (X_{n-1}, X_n)$ are not independent and thus $Z_1, Z_2, \ldots, Z_{n-1}$ are not. The suggestion of Gokhale is applicable to our problem if we are willing to consider values of Z_t coming from disjoint pairs of X_t values, for example, $Z_1 = X_1X_2$, $Z_3 = X_3X_4, \ldots, Z_p = X_pX_{p+1}$ (where p is the biggest odd integer $\leqslant n-1$). Clearly, under H_o, $Z_1, Z_3, \ldots, Z_p$ are independent and standard non-parametric tests for symmetry about zero (the sign test, the Wilcoxon signed rank test, etc.) can be applied to these. Similar tests are also applicable to $Z_2, Z_4, \ldots, Z_q$ (where q is the biggest even integer $\leqslant n-1$), which are also independent under H_o. Nevertheless a test based on $Z_1, Z_3, \ldots, Z_p$ or a test based on $Z_2, Z_4, \ldots, Z_q$ exploits, in a sense, only half of the available sample information and the relation between the two tests would have to be studied. It is certainly desirable to have a test that exploits the full set of the products $Z_1, Z_2, \ldots, Z_{n-1}$.

Let us consider the family of simple linear rank tests for symmetry about zero[4] applied to the variables $Z_t = X_tX_{t+1}$, $t = 1, 2, \ldots, n-1$. These are based on test statistics of the form:

$$S = \sum_{t=1}^{N} u(Z_t)\, a_N(R_t^+) \qquad (2.2)$$

where $N = n-1$ and

$u(.)$ is an indicator function such that

$$u(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \leqslant 0 \end{cases}; \qquad (2.3)$$

R_t^+ = rank of $|Z_t|$ when $|Z_1|, \ldots, |Z_N|$ are ranked in increasing order;

$a_N(.)$ is a score function transforming the ranks R_t^+. The ranks R_t^+ can also be conveniently defined by $R_t^+ = \sum_{i=1}^{N} u(|Z_t| - |Z_i|)$, $t = 1, \ldots, N$;

this formula takes care, in particular, of the cases where there are ties among the $|Z_t|$'s (an event of probability zero)[5]. Under the null hypothesis that $Z_1, \ldots, Z_N$ have pdf's symmetric about zero, the distribution of S is obtained in standard cases with the assumption that the observations $Z_1, \ldots, Z_N$ are independent[6]. The difficulty in the present cas is that, even under H_o, the variables $Z_1, \ldots, Z_N$ are not independent (although they have marginal distributions symmetric about zero). We show, in this paper, that this difficulty is in fact of no consequence here, in the sense that the standard null distributions for S can be used.

We will furthermore use a non-negative function in (2.2):

$$a_N(r) \geqslant 0, \text{ for all } r. \qquad (2.4)$$

Then, under the alternative (1.2) of positive dependence, we expect the number of positive Z_t's to be greater than under H_o, and hence S to take a relatively large value: against this one-sided alternative we use a critical region of the form $\{S > c\}$. Similarly, against the alternative (1.3) of negative dependence, we use a critical region of the form $\{S < c'\}$. Finally, a two-sided test has a critical region of the form $\{S > c \text{ or } S < c'\}$. The critical points c and c' of course depend on the significance levels adopted for the tests.

3. Null Distributions of the Test Statistics

First, let us define the "sign" functions:

$$s(x) = 1, \text{ if } x > 0, \quad = -1, \text{ if } x \leqslant 0, \qquad (3.1)$$

and introduce the following notations:

$$\underline{Z} = (Z_1, \ldots, Z_N)$$
$$|\underline{Z}| = (|Z_1|, \ldots, |Z_N|)$$
$$s(\underline{Z}) = (s(Z_1), \ldots, s(Z_N))$$
$$\underline{R}^+ = (R_1^+, \ldots, R_N^+)$$
$$a_N(\underline{R}^+) = (a_N(R_1^+), \ldots, a_N(R_N^+))$$
$$E = \{1, -1\},\ E^N = \underbrace{E \times E \times \ldots \times E}_{N \text{ times}} \qquad (3.2)$$

where $N = n-1$.

Since we assumed $X_1, \ldots, X_n$ have continuous distributions, the variables $Z_1, \ldots, Z_N$ also have continuous (marginal) distributions. Consequently, the vector $\underline{R}^+$ is with probability 1 a permutation of $(1, 2, \ldots, N)$, and the vector $a_N(\underline{R}^+)$ is a permutation of $(a_N(1), a_N(2), \ldots, a_N(N))$.

The rank statistics $S = \sum_{t=1}^{N} u(Z_t)\, a_N(R_t^+)$ can be viewed as a linear combination of the set of constants $a_N(1), \ldots, a_N(N)$ and thus S can be expressed as[7]

$$S = \sum_{r=1}^{N} T(r)\, a_N(r) \qquad (3.3)$$

where

$$T(r) = 1, \text{ if the r-th smallest observation in absolute value is positive}, \quad = 0, \text{ otherwise}. \qquad (3.4)$$

If the random variables $u(Z_1), \ldots, u(Z_N)$ and the random vector $\underline{R}^+$ are mutually independent with

$$P[u(Z_t) = 0] = P[u(Z_t) = 1] = \tfrac{1}{2},\ t = 1, \ldots, N,$$

the distribution of S is completely determined. This is easy to see by considering the characteristic function of S. Conditional on $\underline{R}^+$, we have (with probability 1):

$$\begin{aligned} E(e^{i\tau S}|\underline{R}^+) &= \prod_{t=1}^{N} E\left[e^{i\tau a_N(R_t^+)\, u(Z_t)}\right] \\ &= \prod_{t=1}^{N}\left\{\tfrac{1}{2}\left[1 + e^{i\tau a_N(R_t^+)}\right]\right\} \\ &= (\tfrac{1}{2})^N \prod_{t=1}^{N}\left[1 + e^{i\tau a_N(r)}\right] \end{aligned} \qquad (3.5)$$

where $i = \sqrt{-1}$. Hence, since $E(e^{i\tau S}|\underline{R}^+)$ does not depend on $\underline{R}^+$, the characteristic function of S is

$$E(e^{i\tau S}) = (\tfrac{1}{2})^N \prod_{r=1}^{N}\left[1 + e^{i\tau a_N(r)}\right]. \qquad (3.6)$$

It is thus entirely determined for a given score function $a_N(.)$ and, consequently, the distribution of S is also determined.

We will now show that, under H_o, (despite the fact that $Z_1, \ldots, Z_N$ are not independent) the random variables $u(Z_1), \ldots, u(Z_N)$ and the random vector $\underline{R}^+$ are mutually independent with

$$P[u(Z_t) = 0] = P[u(Z_t) = 1] = \tfrac{1}{2},\ t = 1, \ldots, N.$$

We first prove the following lemma, which states that the 2^N different "sign" permutations $s(\underline{Z})$ in E^N that can be associated with $\underline{Z} = (Z_1, \ldots, Z_N)$ have the same probability $(\tfrac{1}{2})^N$, conditional on any event $|\underline{Z}| = |\underline{z}|$, provided z is outside a set $A \subseteq R^N$ of probability zero.

LEMMA 3.1: Let $X_1, \ldots, X_n$ be a sequence of independent random variables having pdf's symmetric about zero, and let

$Z_t = X_t X_{t+1}, \quad t = 1, \ldots, n-1.$

Then, using the notations in (3.2),

$$P[s(\underline{Z}) = \underline{s} \mid |\underline{Z}| = |\underline{z}|] = (\tfrac{1}{2})^N, \forall \underline{s} \varepsilon E^N$$

where $N = n-1$, for any $|\underline{z}| = (|z_1|, \ldots, |z_N|)$ such that $|z_t| > 0$, $t = 1, \ldots, N$, and $q(|\underline{z}|) \neq 0$, where $q(.)$ is the pdf of the random vector $|\underline{Z}|$.

PROOF: Since $X_1, \ldots, X_n$ are independent random variables with pdf's symmetric about zero, the pdf of the random vector $(X_1, \ldots, X_n)$ has the property:

$$\begin{aligned} p(x_1, \ldots, x_n) &= \prod_{t=1}^{n} p_{X_t}(x_t) \\ &= \prod_{t=1}^{n} p_{X_t}((-1)^{j_t}|x_t|) \\ &= p((-1)^{j_1}|x_1|, \ldots, (-1)^{j_n}|x_n|) \end{aligned}$$

for all $j_t \varepsilon \{0, 1\}$, $t = 1, \ldots, n$. Assuming $|x_1|, \ldots, |x_n|$ are all different from zero (the set where this does not hold has probability zero), we can say that the event $|X_1| = |x_1|, \ldots, |X_n| = |x_n|$ consists of 2^n different points, or configurations, $(x_1, \ldots, x_n)$ obtained by attaching all possible sign permutations to $|x_1|, \ldots, |x_n|$, and all these configurations have the same probability. Now let us consider the set of configurations $(z_1, \ldots, z_{n-1})$ where

$$z_t = x_t x_{t+1}, \quad t = 1, \ldots, n-1,$$

obtained from the 2^n configurations associated with $|x_1|, \ldots, |x_n|$. Clearly, $z_1, \ldots, z_{n-1}$ are all different from zero. There are 2^{n-1} such different configurations and they all have the same set of absolute values:

$$|z_t| = |x_t||x_{t+1}|, \quad t = 1, \ldots, n-1.$$

Each particular configuration $(z_1, \ldots, z_{n-1})$ in this set is of the form:

$$((-1)^{k_1}|z_1|, \ldots, (-1)^{k_{n-1}}|z_{n-1}|), \quad k_t \varepsilon \{0,1\}, \quad t = 1, \ldots, n-1,$$

and can be obtained from at least 2 among the 2^n basic configurations $(x_1, \ldots, x_n)$: for, if $((-1)^{j_1}|x_n|), \ldots, (-1)^{j_n} x_n|)$ generates $(z_1, \ldots, z_{n-1})$, i.e. if $(z_1, \ldots, z_{n-1}) = ((-1)^{j_1+j_2}|x_1||x_2|, \ldots, (-1)^{j_{n-1}+j_n}|x_{n-1}||x_n|)$ the configuration $((-1)^{j_1+1}|x_1|, \ldots, (-1)^{j_n+1}|x_n|)$. Since there are 2^{n-1} different configurations of the form $((-1)^{k_1}|x_1||x_2|, \ldots, (-1)^{k_{n-1}}|x_{n-1}|x_n|)$ to be generated from 2^n basic ones, we can say that each of them is generated by exactly 2 configurations of the form $((-1)^{j_1}|x_1|, \ldots, (-1)^{j_n}|x_n|)$ (i.e. in the basic set) ans we saw previously that all these have the same probability. Therefore, given $|X_1| = |x_1|, \ldots, |X_n| = |x_n|$, the 2^{n-1} different events in the set

$$A = \{Z_1 = (-1)^{k_1}|z_1|, \ldots, Z_{n-1} = (-1)^{k_{n-1}}|z_{n-1}| : k_t \varepsilon \{0, 1\}, \quad t = 1, \ldots, n-1\}$$

have the same probability. If we integrate over all the vectors $|\underline{x}| = (|x_1|, \ldots, |x_n|)$ such that $|z_t| = |x_t||x_{t+1}|$, $t = 1, \ldots, n-1$ (for $|z_1|, \ldots, |z_{n-1}|$ fixed), a similar partitioning is maintained, hence we can say that the 2^{n-1} different events in the set A have the same probability. (Note that, in this integration, it is sufficient to consider the set of vectors $|\underline{x}|$ having their components $|x_t|$ all different.) Consequently, the pdf of $(Z_1, \ldots, Z_{n-1})$ has the property:

$$q(z_1, \ldots, z_{n-1}) = q((-1)^{k_1}|z_1|, \ldots, (-1)^{k_{n-1}}|z_{n-1}|) \quad \text{for all } k_t \varepsilon \{0, 1\}, \quad t = 1, \ldots, n-1.$$

Hence, given $|Z_1| = |z_1|, \ldots, |Z_{n-1}| = |z_{n-1}|$, all the sign permutations have the same probability. Since there exist 2^{n-1} of these, we have, provided the pdf of $(Z_1, \ldots, Z_{n-1})$ does not vanish at $(z_1, \ldots, z_{n-1})$:

$$P\left[s(Z_1) = (-1)^{k_1}, \ldots, s(Z_{n-1}) = (-1)^{k_{n-1}} \,\middle|\, |Z_1| = |z_1|, \ldots, |Z_{n-1}| = |z_{n-1}|\right] = (\tfrac{1}{2})^{n-1}$$

for all $k_t \varepsilon \{0, 1\}$, $t = 1, \ldots, n-1$; or, which is equivalent,

$$P\left[s(\underline{Z}) = \underline{s} \,\middle|\, |\underline{Z}| = |\underline{z}|\right] = (\tfrac{1}{2})^N, \forall \underline{s} \varepsilon E^N$$

where $N = n-1$. Q.E.D.

The main result then follows.

THEOREM 3.2: Let $X_1, \ldots, X_n$ be a sequence of independent random variables having pdf's symmetric about zero, and let $Z_t = X_t X_{t+1}$, $t = 1, \ldots, n-1$. Then the random variables $u(Z_1), \ldots, u(Z_N)$, where $N = n-1$, and the random vector $\underline{R}^+$ are mutually independent with

$$P[u(Z_t) = 0] = P[u(Z_t) = 1] = \tfrac{1}{2}, \quad t = 1, \ldots, N.$$

PROOF: From Lemma 3.1, we see that

$$P[s(\underline{Z}) = \underline{s} \mid |\underline{Z}| \varepsilon C] = (\tfrac{1}{2})^N, \forall \underline{s} \varepsilon E^N$$

for any set C such that $P(|\underline{Z}| \varepsilon C) \neq 0$. Therefore, $s(\underline{Z})$ is independent of $|\underline{Z}|$ hence also of $\underline{R}^+$. Furthermore, the vector $(s(Z_1), \ldots, s(Z_N))$ has a uniform distribution over the discrete space E^N. Hence, given that exactly 2^{N-1} vectors $(s_1, \ldots, s_N) \varepsilon E^N$ satisfy $s_t = -1$ and the 2^{N-1} other ones satisfy $s_t = 1$, where $1 \leq t \leq N$, we have

$$P[s(Z_t) = -1] = P[s(Z_t) = 1] = \tfrac{1}{2}, \quad t = 1, \ldots, N$$

and the variables $s(Z_1), \ldots, s(Z_N)$ are mutually independent. Finally, if we notice that $u(Z_t)$ is just a one-to-one transformation of $s(Z_t)$ such that

$$u(Z_t) = \begin{cases} 0, & \text{if } s(Z_t) = -1 \\ 1, & \text{if } s(Z_t) = 1 \end{cases},$$

the result follows. Q.E.D.

From theorem 3.2, we conclude that the distribution of the test statistic $S = \sum_{t=1}^{N} u(Z_t) a_N(R_t^+)$ is completely determined under H_0. The mean and variance of S under H_0 are easily computed:

$$E(S) = \tfrac{1}{2} \sum_{t=1}^{N} a_N(t), \tag{3.7}$$

$$Var(S) = \tfrac{1}{4} \sum_{t=1}^{N} a_N^2(t). \tag{3.8}$$

Furthermore, the distribution of S is symmetric about E(S) and approximately normal for

$$\max_{1 \leq r \leq N} \frac{a_N^2(r)}{\sum_{t=1}^{N} a_N^2(t)} \text{ sufficiently small}^{8}.$$

We will now examine special cases of the family of tests represented by S by specifying the score function $a_N(.)$.

4. An Exact Runs Test

If we consider the constant score function $a_N(r) = 1$, the statistic S takes the form:

$$S = \sum_{t=1}^{N} u(Z_t). \tag{4.1}$$

S is then the number of positive values in the sequence $Z_1, \ldots, Z_N$, i.e. the statistic of the sign test applied to $Z_1, \ldots, Z_N$. Assuming $X_1, \ldots, X_n$ are all different from zero (an event of probability 1), S is also the number of times consecutive X_t's have the same sign. Under H_0, $S \sim Bi(N, \tfrac{1}{2})$, i.e. S follows a binomial distribution with number of trials N and probability of "success" $\tfrac{1}{2}$. The fact that S takes a relatively big value ($S > c$) can be considered as evidence of positive serial dependence and the proper critical point c for a test of level α can easily be found. Similarly, if S takes too small a value ($S < c'$), this can be viewed as evidence of negative dependence between consecutive X_t's.

Again, assuming $X_1, \ldots, X_n$ are all different from zero, N-S is the number of times changes of sign occur in the sequence $X_1, \ldots, X_n$; hence $N-S+1 = n-S$ is the total number of runs in the sequence of the signs $s(X_1), \ldots, s(X_n)$, where

$$s(x) = + \text{, if } x > 0,$$
$$= - \text{, if } x < 0.$$

Under H_o, $N-S \sim Bi(N,\frac{1}{2})$, since $S \sim Bi(N,\frac{1}{2})$. Therefore, the null distribution of the total number of runs $R = n - S$ can be characterized by

$$R - 1 \sim Bi(n-1,\tfrac{1}{2}), \qquad (4.2)$$

hence the mean and the variance of R are:

$$E(R)=1+\frac{n-1}{2}=\frac{n+1}{2}, \quad Var(R) = \frac{n-1}{4}$$

Too small a number of runs ($R < n - c$) points out to the presence of positive serial dependence while a too big number ($R > n + c'$) points out to negative serial dependence.

Other tests of randomness based on the total number of runs using the same sequence of signs were described by David (1947), Goodman (1958) and Granger (1963). David's test differs from the one presented above in the sense that the null distribution of the number of runs is obtained conditionally on the number of +'s and the number of -'s in the sequence. As regards Goodman and Granger, they considered only large sample tests.

5. The Signed-Rank Test and Other Tests

If we take a $a_N(r) = r$, the statistic S becomes

$$S = \sum_{t=1}^{N} u(Z_t)\, R_t^+. \qquad (5.1)$$

S is the sum of the ranks of the positive Z_t's. It is the test statistic associated with the signed-rank test for symmetry about zero, or Wilcoxon test, when applied to $Z_1, \ldots, Z_N$. The distribution of S, under H_o, is exactly the same as the null distribution of the Wilcoxon test statistic (as can be seen easily by comparing with Lehmann 1975, p. 165). It is very extensively tabulated in Wilcoxon, Katt and Wilcox (1968). To test H_o against the alternative (1.2), we use a one-sided critical region of the form $\{S > c\}$; similarly, against the alternative (1.3), we use $\{S < c'\}$. A two-sided test would use a region of the form $\{S > c \text{ or } S < c'\}$. This test does not use only information about the signs of the random variables $Z_1, \ldots, Z_N$, like the runs test of section 4, but also information regarding their sizes via the ranks $R_1^+, \ldots, R_N^+$. In a wide class of cases, it can thus be expected to have greater power than the runs test.

Other tests can be generated by using other score functions. We will mention the analogues of two other well-known tests of symmetry. If we take the scores

$$a_N(r) = E|V|^{(r)}, \qquad (5.2)$$

where $|V|^{(r)}$ is the r-th order statistic from the absolute values of a normal N(0, 1) random sample, we get an analogue of the Fraser test (also called normal scores test). If we take

$$a_N(r) = \Phi^{-1}\left(\tfrac{1}{2} + \tfrac{1}{2}\,\frac{r}{N+1}\right), \qquad (5.3)$$

where $\Phi(.)$ is the cumulative distribution function of a N(0, 1) random variable and $\Phi^{-1}(.)$ is the inverse function of $\Phi(.)$, we get an analogue of the van der Waerden test. For further details, the reader is referred to Hájek and Šidák (1967, pp. 108-111).

6. Generalization

The tests of independence we considered in the preceding sections are based on considering the products between values of X_t separated by a lag of only one period. Analogous tests can be performed by considering sequences of products between values of X_t separated by lags greater than 1:

$$Z_t = X_t X_{t+k}, \quad t = 1, \ldots, n-k, \qquad (6.1)$$

where $k > 1$, and test statistics of the form:

$$S = \sum_{t=1}^{N} u(Z_t)\, a_N(R_t^+), \qquad (6.2)$$

where $N = n - k$ and the ranks R_t^+, $t = 1, \ldots, N$, are defined with respect to the sequence $|Z_t|$, $t = 1, \ldots, N$. The corresponding tests can be run against alternatives of the type:

$$Med(X_t X_{t+k}) > 0, \quad t = 1, \ldots, n-k \qquad (6.3)$$

or

$$Med(X_t X_{t+k}) < 0, \quad t = 1, \ldots, n-k. \qquad (6.4)$$

All that is required for doing this is to generalize Lemma 3.1 and Theorem 3.2 in order to make them applicable to the sequences defined by (6.1). This is straightforward but, for completeness, we state the generalized results.

LEMMA 6.1: Let $X_1, \ldots, X_n$ be a sequence of independent random variables having pdf's symmetric about zero, and let

$$Z_t = X_t X_{t+k}, \quad t = 1, \ldots, n-k,$$

where $k \geq 1$. Then, using the notations in (3.2),

$$P\left[s(\underline{Z}) = \underline{s} \mid |\underline{Z}| = |\underline{z}|\right] = (\tfrac{1}{2})^N, \forall\, \underline{s} \varepsilon E^N$$

where $N = n - k$, for any $|\underline{z}| = (|z_1|, \ldots, |z_N|)$ such that $|z_t| > 0$, $t = 1, \ldots, N$, and $q(|\underline{z}|) \neq 0$, where $q(.)$ is the pdf of the random vector $|\underline{Z}|$.

PROOF: For $k = 1$, this is identical with Lemma 3.1. For $k > 1$, the proof is exactly similar except that each set A of 2^n different configurations

$$((-1)^{j_1}|x_1|, \ldots, (-1)^{j_n}|x_n|),\ j_t \varepsilon \{0, 1\}, \quad t = 1, \ldots, n$$

(and where $x_1 \ldots, x_n$ are all different from zero) generates a set B of 2^{n-k} different configurations of the form $((-1)^{k_1}|z_1|, \ldots, (-1)^{k_N}|z_N|)$, where $k_t \varepsilon \{0, 1\}$, $t = 1, \ldots, N$, with $N = n - k$, and $|z_t| = |x_t||x_{t+k}|$, $t = 1, \ldots, N$. Then, each of these can be seen to be generated by exactly 2^k configurations in A, hence they have equal probabilities (conditional on B). The result then follows in the same way it does in Lemma 3.1. Q.E.D.

THEOREM 6.2: Let $X_1, \ldots, X_n$ be a sequence of independent random variables having pdf's symmetric about zero, and let

$$Z_t = X_t X_{t+k}, \quad t = 1, \ldots, n-k,$$

where $k > 1$. Then random variables $u(Z_1), \ldots, u(Z_N)$, where $N = n-k$, and the random vector $\underline{R}^+$ are mutually independent with

$$P[u(Z_t) = 0] = P[u(Z_t) = 1] = \tfrac{1}{2}, \quad t = 1, \ldots, N.$$

PROOF: Identical to the proof of Theorem 3.2, except that Lemma 6.1 is used in place of Lemma 3.1. Q.E.D.

7. Concluding Remarks

We already stressed the fact that the tests described above are exact and do not require the standard assunption of normality. It is also worthwhile to note that the process $\{X_t\}_{t=1}^{T}$ does not have to be stationary; in particular, the variances (if they exist) of the variables X_t may differ under H_o, without the null distributions of the test statistics' being affected. Furthermore, we can see easily that the alternatives (1.2)-(1.3) and (6.3)-(6.4) are very wide, including covariance stationary schemes as well as a number of non-stationary ones.

The main limitations are the assumptions of symmetry and known (or zero) median. In practice, what we test is the joint hypothesis that the X_t's are independent and have the pdf's symmetric about zero (hence, if second moments exist, $E(X_t X_{t+k}) = 0$, $t = 1, \ldots, n-k$, for any k). The symmetry assumption may reasonably be maintained in a large number of situations, but it would certainly be desirable (althought possibly not easy if exactness is to be preserved) to be able to relax the assumption of known median.

Finally, although the main area of applicability of the proposed tests is the analysis of time series data, it should be pointed out that they can be used on any set of random variables meeting the same assumptions.

APPENDIX

The following concepts of dependence (or association) between random variables were developed by Lehmann (1966) and extended by Jogdeo (1968).

A pair (X, Y) of random variables are said to be "positively quadrant dependent" if

$$P[X \leq x, Y \leq y] \geq P[X \leq x]P[Y \leq y], \quad \text{for any } x, y. \quad (A1)$$

This dependence may be called "strict" at (x_0, y_0) if a strict inequality between the two sides of (A1) holds for $(x, y) = (x_0, y_0)$. Assuming $P[X \leq x] > 0$, the condition (A1) can be rewritten:

$$P[Y \leq y | X \leq x] \geq P[Y \leq y] , \quad (A2)$$

which shows clearly that the knowledge that X is small increases the probability of Y being small. A special case where (A1) holds is the case where

$$P[Y \leq y | X = x] \text{ is non-increasing in } x, \quad (A3)$$

designated by Lehmann (1966) as "positive regression dependence" of Y on X, a condition probably closer to the intuitive notion of "positive dependence". When X and Y have finite second moments, the condition (A1) can be shown to imply:

$$E(XY) \geq E(X)E(Y) \quad (A4)$$

(see Lehmann 1966, Lemma 2), which means that X and Y have a non-negative covariance.

Similarly, "negative quadrant dependence" is defined by:

$$P[X \leq x, Y \leq y] \leq P[X \leq x]P[Y \leq y], \quad \text{for any } x, y, \quad (A5)$$

and "negative regression dependence" of Y on X by the condition:

$$P[Y \leq y | X = x] \text{ is non-decreasing in } x; \quad (A6)$$

further, (A4) takes in this case the form:

$$E(XY) \leq E(X)E(Y). \quad (A7)$$

Now let us assume the random variables X and Y have continuous (marginal) distribution symmetric about zero; then

$$P[X \geq 0] = P[X \leq 0] = P[Y \geq 0] = P[Y \leq 0] = \tfrac{1}{2}. \quad (A8)$$

Consider the product Z = XY. If X and Y are independent, it is easy to see that Z has a distribution symmetric about zero, hence

$$P[Z \geq 0] = P[Z \leq 0] = \tfrac{1}{2}. \quad (A9)$$

If X and Y are "strictly positively quadrant dependent" for any x, y, we have:

$$P[Y \leq 0 | X \leq 0] > P[Y \leq 0] = \tfrac{1}{2}. \quad (A10)$$

The inequality (A10) also implies (see Lehmann 1966, Lemma 1):

$$P[Y \geq 0 | X \geq 0] > P[Y \geq 0] = \tfrac{1}{2}. \quad (A11)$$

Then, from (A8), (A10) and (A11), we get:

$$P[Z \geq 0] = P[Y \geq 0 | X \geq 0]P[X \geq 0] + P[Y \leq 0 | X \leq 0]P[X \leq 0]$$

$$= \tfrac{1}{2}\{P[Y \geq 0 | X \geq 0] + P[Y \leq 0 | X \leq 0]\} > \tfrac{1}{2} \cdot \quad (A12)$$

Similarly, if X and Y are "strictly negatively quadrant dependent" for any x, y, we have:

$$P[Z \leq 0] < \tfrac{1}{2}. \quad (A13)$$

The relationships (A9), (A12) and (A13) show clearly the link between symmetry about zero of the product XY and the independence of X and Y.

FOOTNOTES

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[1] Alternatively, we could assume $X_1, \ldots, X_n$ have known medians $\mu_1, \ldots, \mu_n$ and then, set the medians of the observation to zero by considering $X_1 - \mu_1, \ldots, X_n - \mu_n$.

[2] To be more general, we could also use the signs $\geq$ and $\leq$ in (1.2) and (1.3) respectively, stating that at least one strict inequality must hold in each case.

[3] See Johnston 1972, pp. 305-307, Box and Jenkins 1970, Chs 2, 6.

[4] See Hájek 1969, Ch. 5.

[5] For further discussion of the treatment of ties, see Hájek 1969, Ch. VII.

[6] See Hájek 1969, Ch. 5.

[7] See Hájek 1969, p. 106.

[8] See Hájek 1969, p. 106.

REFERENCES

Blattberg, R., and Gonedes, N. (1974), "A comparison of the stable and Student distributions as statistical models for stock prices". Journal of Business, 47, 244-280.

Box, G.E.P., and Jenkins, G.M. (1970), Time Series Analysis Forecasting and Control, Holden-Day, San Francisco.

David, F.N. (1947), "A power function for tests of randomness in a sequence of alternatives" Biometrika, 34, 335-339.

Fama, E.F. (1965), "The behaviour of stock market prices". Journal of Business, 38, 34-105.

Gokhale, D.V. (1970), "On symmetry of the product of two random variables and their independence". Sankhya A, 32, 329-336.

Goodman, L. (1958), "Simplified runs tests and likelihood ratio tests for Markov chains". Biometrika, 45, 181-197.

Granger, C.W.J. (1963), "A quick test for serial correlation suitable for use with non-stationary time series". JASA, 58, 728-736.

Hájek, J. (1969), A Course in Nonparametric Statistics. Holden-Day, San Francisco.

Hájek, J., and Šidák, Z. (1967), Theory of Rank Tests. Academic Press, New York and London.

Jogdeo, K. (1968), "Characterizations of independence in certain families of bivariate and multivariate distributions". Ann. Math. Statist., 39, 433-441.

Johnston, J. (1972), Econometric Methods, 2nd Edition. McGraw-Hill, New York.

Lehmann, E.L. (1966), "Some concepts of dependence". Ann. Math. Statist., 37, 1137-1153.

(1975), Nonparametrics. Statistical Methods Based on Ranks. Holden-Day, San Francisco.

Mandelbrot, B.(1963), "The variation of certain speculative prices". Journal of Business, 36, 394-419.

Moore, G.H., and Wallis, W.A. (1943), "Time series significance tests based on signs of differences". JASA, 38, 153-164.

Praetz, P.D. (1972), "The distribution of share price changes". Journal of Business, 45, 49-55.

Rao, C.R. (1973), Linear Statistical Inference and Its Applications, Second Edition, Wiley New York.

Stuart, A. (1952), "The power of two difference-sign tests". JASA, 47, 416-424.

Wald, A., and Wolfowitz, J. (1943), "An exact test for randomness in the non-parametric case based on serial correlation". Ann. Math. Statist., 14, 378-388.

Wallis, W.A., and Moore, G.H. (1941), "A significance test for time series analysis". JASA, 36, 401-409.

Wilcoxon, F., Katti, S.K., and Wilcox, R.A. (1968), "Critical values and probability levels for the Wilcoxon rank sum test and the Wilcoxon signed rank test", In Selected Tables in Mathematical Statistics, Volume I, Edited by the Institute of Mathematical Statistics (1973), 171-259.